SymTFTs and Duality Defects from 6d SCFTs on 4-manifolds

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Introduction

Motivation

6D SCFTs are strongly coupled theories. The compactification of them gives rise to many interesting lower dimensional SCFTs. For example,

- Riemann surface to Class S theories. [Gaiotto 09']...
- hyperbolic 3-manifolds to 3d $\mathcal{N} = 2$ SCFTs. [Dimofte, Gaiotto, Gukov 11']...
- 4-manifolds gives 2d $\mathcal{N} = (0, 2)$ SCFT [Gadde1, Gukov,Putrov 13']...

Useful to understand the dualities and correspondences of them.

Consider the 2d SCFTs $T_N[M_4]$ from the compactification of 6d $\mathcal{N} = (2, 0)$ theory of A_{N-1} type on 4-manifold M_4. Very little is known about $T_N[M_4]$.

- Non-Lagrangian
- Less supersymmetry

Generalized symmetries

Global symmetries are generated by topological defects. [Gaiotto, Kapustin, Seiberg, Willett 14']...

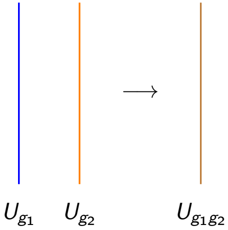
- Higher-form symmetry [Gaiotto, Kapustin, Seiberg, Willett 14']...
- Higher-group symmetry [Kapustin,Thorngren 13', Cordova,Dumitrescu,Intriligator 18', Benini,Cordova,Hsin 18',. . .]
- Subsystem symmetry [Paramekanti, Balents, Fisher 02', Lawler Fradkin 04',Seiberg,Shao 20', Cao, Li, Yamazaki, Zheng 23']...
- Non-invertible (categorical) symmetry [Verlinde 88', Bhardwaj,Tachikawa 17', Chang,Lin,Shao,Wang,Yin 18',. . .]
- Groupiod symmetry? [Xinyu Zhang's talk]
- . . .

Motivation: understand the (discrete) symmetries of $T_N[M_4]$!

Invertible symmetries

- In d-dimensional QFTs, the p-form symmetries $G^{(p)}$ are generated by codimensional-(p+1) topological defects $U_g(\Sigma_{d-p-1})$, $g \in G$.
- The fusion rule is

$$U_{g1}(\Sigma_{d-p-1})U_{g_2}(\Sigma_{d-p-1}) = U_{g_1 g_2}(\Sigma_{d-p-1})$$



Non-invertible symmetries

Non-invertible symmetries are described by the (higher) fusion category. The fusion rule is not group, but more general as

$$U_a U_b = \sum_c N_{ab}^c U_c$$

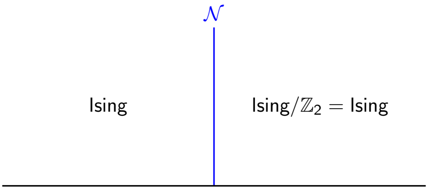
The non-invertible symmetries are very common in 2d RCFT. There they are generated by topological defect lines (TDL) [Jin Chen's talk]. For example, the Ising CFT has three TDLs $\{1, \eta, \mathcal{N}\}$. The fusion rule is

$$\eta \times \eta = 1, \quad \eta \times \mathcal{N} = \mathcal{N}, \quad \mathcal{N} \times \mathcal{N} = 1 + \eta$$

Here $\mathcal{N}$ is the Kramers-Wannier duality defect [Kramers,Wannier, 41']. Symmetries with the this fusion rule is described by the Tambara-Yamagami category $TY(\mathbb{Z}_2)$ [Tambara,Yamagami, 98'].

Duality defect

The Ising CFT is self-dual under gauging $\mathbb{Z}_2$



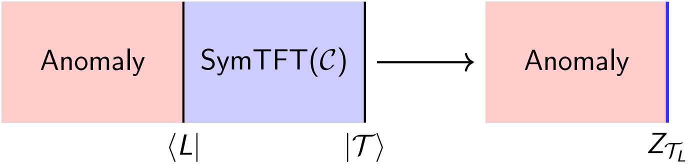
Found in many $d > 2$ QFTs:

- gauging a 0-form symmetry with mixed anomalies with higher-form symmetry [Kaidi, Ohmori, Zheng 21]...
- gauging higher-form symmetry in the half spacetime [Choi, Cordova, Hsin, Lam, Shao 21 & 22]...
- gauging a 0-form symmetry that acts on a higher-form symmetry [Bhardwaj, Bottini, Schafer-Nameki, Tiwari 22]...

SymTFT

Symmetry TFT (SymTFT) of a d-dimensional QFT with symmetry $\mathcal{C}$ is a (d+1)-dimensional TQFT encodes [Apruzzi, Bonetti, Etxebarria, Hosseini, Schafer-Nameki 22]...

- Global variants
  Eg: 4d $\mathcal{N} = 4$ SYM with gauge algebra A_1 has $SU(2)$, $SO(3)_+$ and $SO(3)_-$.
- Symmetries and possible t'hooft anomaly



6d SCFTs are naturally fit in this picture and studied using SymTFT.

Outline

1. Introduction

2. 6d SCFTs on 4-manifolds
2.1 SymTFT
2.2 Global variants
2.3 Duality defects

3. Example
3.1 $\mathbb{P}^1 \times \mathbb{P}^1$
3.2 Connected sum of $\mathbb{P}^1 \times \mathbb{P}^1$
3.3 Hirzebruch surface

4. Conclusion

6d SCFTs on 4-manifolds

6d SCFTs as relative theories

- 6d SCFTs with non-trivial defect group are relative [Freed, Teleman '12]... i.e. they are living on the boundary of a non-invertible TQFT. For 6d $\mathcal{N} = (2,0)$ SCFT of type A_{N-1}, the defect group $\mathcal{D} = \mathbb{Z}_N$ and the 7d TQFT is [Witten, 97']...

$$S_{7d} = \frac{N}{4\pi} \int_{W_7} c \wedge dc \ ,$$

- To make it absolute, one needs to specify the maximal isotropic sublattice (polarization) in $\mathcal{L} \subset H_3(M_6, \mathbb{Z}_N)$ [Tachikawa 13',Gukov, Hsin, Pei, 21'],...

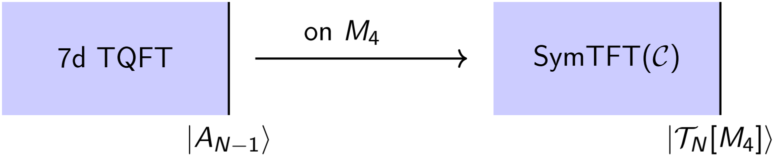
$$\langle M_3, M_3' \rangle = 0 \ , \qquad \forall \ M_3, M_3' \in \mathcal{L} \, .$$

where $\langle -, - \rangle$ is the pairing

$$H_3(M_6, \mathbb{Z}_N) \otimes H_3(M_6, \mathbb{Z}_N) \to \mathbb{Z}_N \ .$$

Compactification on 4-manifolds

- Compactification of 7d/6d coupled system on M_4 leads to 3d/2d coupled system.



- After the twisted compactification, $T_N[M_4]$ is expected to be a 2d $\mathcal{N}=(0,2)$ theory with central charge [Alday, Benini, Tachikawa 09',Gadde, Gukov, Putrov 13']...

$$c_L = \chi(N-1) + (2\chi + 3\sigma)N(N^2-1),$$
$$c_R = \frac{3}{2}(\chi+\sigma)(N-1) + (2\chi+3\sigma)N(N^2-1)$$

Reduction of the 7d TQFT

Consider the 4-manifold M_4 with the cohomology

$$H^*(M_4, \mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}^{b_1}, \mathbb{Z}^{b_2} \oplus \bigoplus_\alpha \mathbb{Z}_{l_\alpha}, \mathbb{Z}^{b_1} \oplus \bigoplus_\alpha \mathbb{Z}_{l_\alpha}, \mathbb{Z}) \,.$$

By differential cohomology class, the 7d TQFT is [Apruzzi, Bonetti, Etxebarria, Hosseini, Schafer-Nameki 22',Beest,Gould,Schäfer-Nameki, Wang 22']...

$$S_{7\mathrm{d}} = \frac{N}{4\pi} \int \breve{G}_4 \star \breve{G}_4$$

Expand the $\breve{G}_4$ as

$$\breve{G}_4 = \sum_{i=1}^{b_1} \breve{F}_3^i \star \breve{v}_1^i + \sum_{i=1}^{b_2} \breve{F}_2^i \star \breve{v}_2^i + \sum_{i=1}^{b_1} \breve{F}_1^i \star \breve{v}_3^i + \sum_\alpha \breve{B}_1^\alpha \star \breve{t}_3^\alpha + \sum_\alpha \breve{B}_2^\alpha \star \breve{t}_2^\alpha \,.$$

where v_n^i and t_n^α are the generators of the free/torsional cycles.

SymTFT

The most general 3d SymTFT is

$$S_{3d} = \frac{N}{4\pi}\sum_{i,j=1}^{b_2} Q^{ij}\int_{W_3} a^i \wedge da^j + \frac{N}{2\pi}\sum_{i,j=1}^{b_1}\left(\int_{M_4}\breve{v}_1^i \star \breve{v}_3^j\right)\int_{W_3} c_0^i \wedge db^j$$
$$+ \frac{N}{4\pi}\sum_{\alpha,\beta}\left(\int_{M_4}\breve{t}_2^\alpha \star \breve{t}_3^\beta\right)\int_{W_3}\breve{B}_2^\alpha \star \breve{B}_1^\beta \,.$$

where $F_2^i = da^i$, $F_1^i = dc_0^i$ and $F_3^i = db^i$, and Q is the intersection form of homology lattice:

$$Q_{M_4} : H_2(M_4,\mathbb{Z}) \otimes H_2(M_4,\mathbb{Z}) \to \mathbb{Z}\,.$$

For a given basis, Q is a $b_2 \times b_2$ matrix. In the following, we will only consider M_4 without 1/3 cycles and torsional cycles.

- Simplified SymTFT is

$$S_{3d} = \frac{1}{4\pi} \sum_{ij} K^{ij} \int_{W_3} a_i \wedge da_j, \quad i,j = 1,2,\ldots,b_2$$

  with Chern-Simons level matrix $K^{ij} = NQ^{ij}$.
- Discriminate group is $\mathcal{D} = H_2(M_4, \mathbb{Z}_N)$. The anyons are labeled by $\vec{\alpha} \in H_2(M_4, \mathbb{Z}_N)$ [Belov, Moore 05']....
- Topological spin:

$$\theta(\vec{\alpha}) \equiv \exp\left[\pi i \vec{\alpha}^t K^{-1} \vec{\alpha}\right]$$

- Braiding:

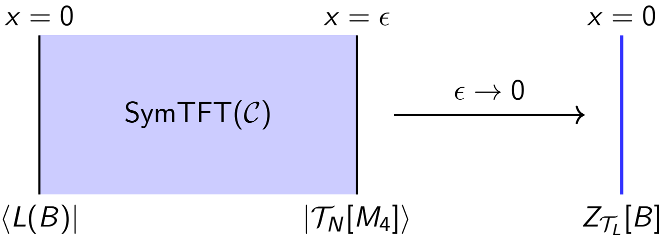
$$B(\vec{\alpha}, \vec{\beta}) \equiv \frac{\theta(\vec{\alpha} + \vec{\beta})}{\theta(\vec{\alpha})\theta(\vec{\beta})} = \exp\left[2\pi i \vec{\alpha}^t K^{-1} \vec{\beta}\right],$$

Absolute theory: topological boundary conditions

- Topological boundary condition $L \subset H_2(M_4, \mathbb{Z}_N)$ such that

$$B(\vec{\alpha}, \vec{\beta}) = 1 \ , \qquad \vec{\alpha}, \vec{\beta} \in L$$

- Absolute theories $\leftrightarrow$ topological boundary conditions [Kaidi, Ohmori, Zheng 22]....



- Symmetries $\leftrightarrow$ $\mathcal{L}^{\perp} = H_2(M_4, \mathbb{Z}_N)/\mathcal{L}$ [Gukov, Hsin, Pei, 21'],...
- t'hooft anomalies: braidings between lines in $\mathcal{L}^{\perp}$ [Kaidi, Nardoni, Zafrir, Zheng, 23']...

Global variants

- Global variants: absolute theories with different SPT phase

$$Z_{\mathcal{T}}[A, L] = Z_{\mathcal{T}}[A, L]\nu(A), \quad \nu(A) \in H^2(G, U(1))$$

- Automorphisms of the SymTFT

$$\begin{aligned} \mathrm{Aut}_G(Q) &= \left\{ T \in GL(b_2, G) \middle| T^t Q T = Q \right\} \\ &= \mathrm{Aut}(G) \times \mathcal{O}_G(Q) \end{aligned}$$

- Topological manipulations $\leftrightarrow$ Generator of $\mathcal{O}_G(Q)$
- Global variants $\leftrightarrow$ $\mathcal{O}_G(Q)$ and the number of the global forms is

$$d(N) = \frac{|\mathrm{Aut}_G(Q)|}{|\mathrm{Aut}(G)|} = |\mathcal{O}_G(Q)|$$

In the following, we set $G = \mathbb{Z}_N$.

Dualities

- The $SL(2,\mathbb{Z})$ duality in 4d $\mathcal{N}=4$ SYM can be tracked from the mapping class group of T^2. [Witten 95]...

$$\mathrm{MCG}(T_2) = \left\{P \in GL(2,\mathbb{Z}) | P^t JP = J\right\},$$

where J is the standard sympletic form

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- Mapping class group of M_4 is

$$\mathrm{MCG}(M_4) = \left\{P \in GL(b_2,\mathbb{Z}) | P^t QP = Q\right\}$$

It gives the duality group of the $T_N[M_4]$. In the basis of the $H_2(M_4,\mathbb{Z}_N)$, we will find the global variants transform under these dualities $\mathrm{MCG}(M_4)$.

Representation of global variants

One can associate each global variant with a dimensional-b_2 matrix $M \in \mathcal{O}_N(Q)$ in $GL(b_2, \mathbb{Z}_N)$.

- Choose the rep of the generator of $\mathcal{O}_N(Q)$.
- Identify one of the global variants as the identity matrix. The reps of all others are fixed by matrix multiplication.
- action of topological manipulation

$$M \to MG, \qquad G \in \mathrm{Aut}_{\mathbb{Z}_N}(Q) \ .$$

- action of duality

$$M \to F^t M, \qquad F \in \mathrm{MCG}(M_4),$$

Given any SymTFT, one can obtain the web of global variants of $T_N[M_4]$ transforming between each other by dualities and topological manipulations.

Example

Reduction of 7d TQFT

- Homology of $\mathbb{P}^1 \times \mathbb{P}^1$:

$$H_*(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) = \{\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}\},$$

- Let b and f be a basis of $H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$, with intersection numbers

$$b^2 = 0\ ,\ \ f^2 = 0\ ,\ \ b \cdot f = 1\,.$$

- Reduction with

$$a = \int_b c, \qquad\qquad \hat{a} = \int_f c\ .$$

The 3d action is

$$S_{3d} \ \ = \ \ \frac{N}{2\pi} \int_{W_3} a \cup \delta\widehat{a}\,,$$

$\mathbb{Z}_N$ gauge theory

- The 3d $\mathbb{Z}_N$ discrete gauge theory has N^2 line operators,

$$L_{(e,m)}(\gamma) = \exp\left(\frac{2\pi i}{N}\oint_\gamma ea\right)\exp\left(\frac{2\pi i}{N}\oint_\gamma m\hat{a}\right) \ ,$$

where $(e, m) \in \mathbb{Z}_N \times \mathbb{Z}_N$ are the charges.

- The braiding between them is

$$L_{(e,m)}(\gamma)L_{(e',m')}(\gamma') = e^{-\frac{2\pi i}{N}(em'+me')\langle\gamma,\gamma'\rangle}L_{(e',m')}(\gamma')L_{(e,m)}(\gamma) \ ,$$

where $\langle\gamma, \gamma'\rangle$ represents the intersection number on Σ_2.

Operations

- τ: stacking Arf invariants

$$Z_{\tau\mathcal{T}}[\eta] = (-1)^{\mathrm{Arf}(\eta)} Z_{\mathcal{T}}[\eta] \ ,$$

- σ: gauging 0-form symmetry $\mathbb{Z}_N$

$$Z_{\sigma\mathcal{T}}[A] = \frac{1}{|H^0(\Sigma_2, \mathbb{Z}_N)|} \sum_{a \in H^1(\Sigma_2, \mathbb{Z}_N)} Z_T[a]\, e^{\frac{2\pi i}{N} \int_{\Sigma_2} a \cup A} \ ,$$

- s: 2d duality from $\mathrm{MCG}(\mathbb{P}^1 \times \mathbb{P}^1)$

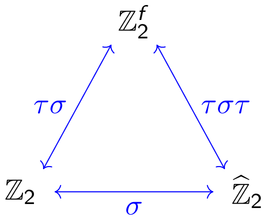
$$\mathbb{P}^1_A \xleftrightarrow{\ s\ } \mathbb{P}^1_B, \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Absolute theories with $N = 2$

There are 3 topological boundary conditions

$$\begin{aligned} L_1 &= \{(0,0), \quad (0,1)\} &\rightarrow& \quad \mathbb{Z}_2 \\ L_2 &= \{(0,0), \quad (1,0)\} &\rightarrow& \quad \widehat{\mathbb{Z}}_2 \\ L_3 &= \{(0,0), \quad (1,1)\} &\rightarrow& \quad \mathbb{Z}_2^f \end{aligned}$$

These are the groupoid orbifold studied in [Gaiotto Kulp 20']...



Global variants with $N = 2$

- Automorphism group is $\text{Aut}_{\mathbb{Z}_2}(Q) = S_3$ with

$$\mathcal{O}_2(Q) = S_3 \,, \qquad \text{Aut}(\mathbb{Z}_2) = 1$$

- There are 6 global variants denote them by

$$(\mathbb{Z}_2)_i, \quad (\widehat{\mathbb{Z}}_2)_i, \quad (\mathbb{Z}_2^f)_i, \quad i = 0, 1$$

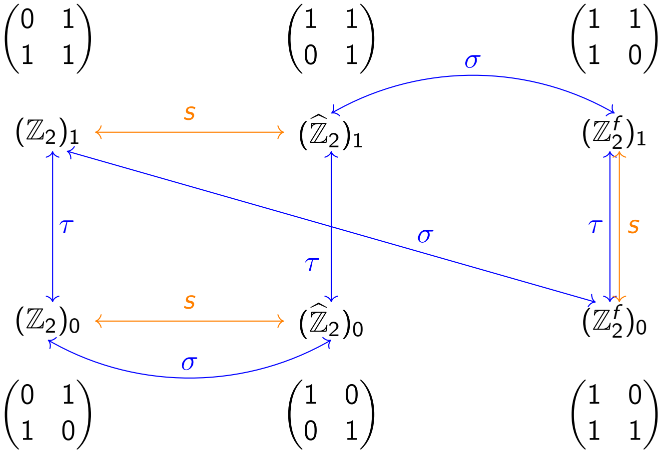
- Label them by the two dimensional representation of S_3 by assigning

$$M[(\widehat{\mathbb{Z}}_2)_0] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the generators of S_3

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Web of global variants



Absolute theories with prime $N = p > 2$

- There are two topological boundary conditions

$$L_1 = \{(0,0),(1,0),\ldots,(p-1,0)\} \quad \rightarrow \quad \mathbb{Z}_p$$
$$L_2 = \{(0,0),(0,1),\ldots,(0,p-1)\} \quad \rightarrow \quad \widehat{\mathbb{Z}}_p$$

- Automorphism group $\mathrm{Aut}_{\mathbb{Z}_p}(Q)$ is the Dihedral group

$$D_{2(p-1)} = \langle r, \sigma | r^{p-1} = \sigma^2 = (\sigma r)^2 = 1 \rangle$$

equivalently as $\mathbb{Z}_p^\times \ltimes \mathbb{Z}_2$ with

$$\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^\times = \langle r \rangle, \qquad \mathcal{O}_p(Q) = \mathbb{Z}_2 = \langle \sigma \rangle.$$

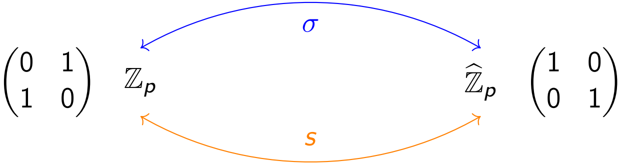
Note that $\mathrm{Aut}(\mathbb{Z}_p)$ represents different ways to choose generator in $\mathrm{Aut}(\mathbb{Z}_p)$ or insert $\mathbb{Z}_p$ TDLs does not give rise to new global variants.

Global variants with prime $N = p > 2$

- Since $\mathbb{Z}_p$ does not have $\mathbb{Z}_2$ subgroup, there is no way to stack SPT phase, there are only two global variants. We label them by specifying the following assignment

$$M[\widehat{\mathbb{Z}}_p] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- Web of global variants



Absolute theories with $N = 4$

- Topological boundary conditions:

$$\begin{array}{lllllll} L_1 = \{(0,0), & (0,1), & (0,2), & (0,3)\} & \to & \mathbb{Z}_4 \\ L_2 = \{(0,0), & (1,0), & (2,0), & (3,0)\} & \to & \widehat{\mathbb{Z}}_4 \\ L_3 = \{(0,0), & (0,2), & (2,1), & (2,3)\} & \to & \mathbb{Z}_4^f \\ L_4 = \{(0,0), & (2,0), & (1,2), & (3,2)\} & \to & \widehat{\mathbb{Z}}_4^f \\ L_5 = \{(0,0), & (0,2), & (2,0), & (2,2)\} & \to & (\mathbb{Z}_2 \times \widehat{\mathbb{Z}}_2)_{\mu_3} \end{array}$$

- The automorphism group is $\text{Aut}_{\mathbb{Z}_4}(Q) = \mathbb{Z}_2 \times D_8$ with

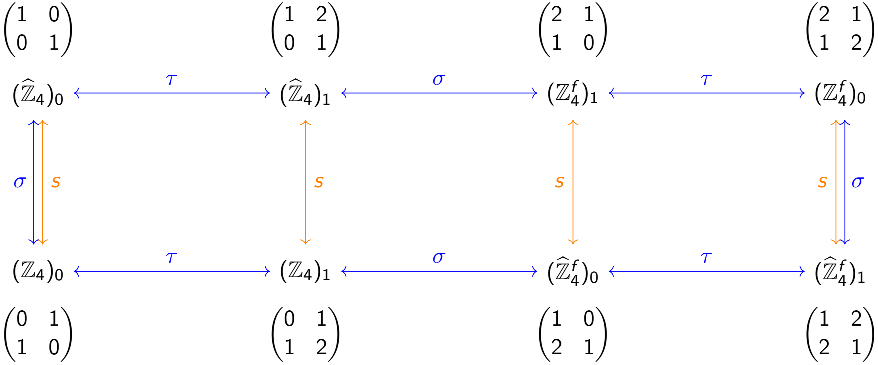
$$\mathcal{O}_4(Q) = D_8 \ , \qquad \text{Aut}(\mathbb{Z}_4) = \mathbb{Z}_2$$

- There are 8 global forms transforming into each other by gauging $\mathbb{Z}_4$ and stacking Arf.

Web of global variants

The 2-dimensional rep of global variants are given below

$$M[(\widehat{\mathbb{Z}}_4)_0] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$



Global variants with $N = 6$

- Topological boundary conditions:

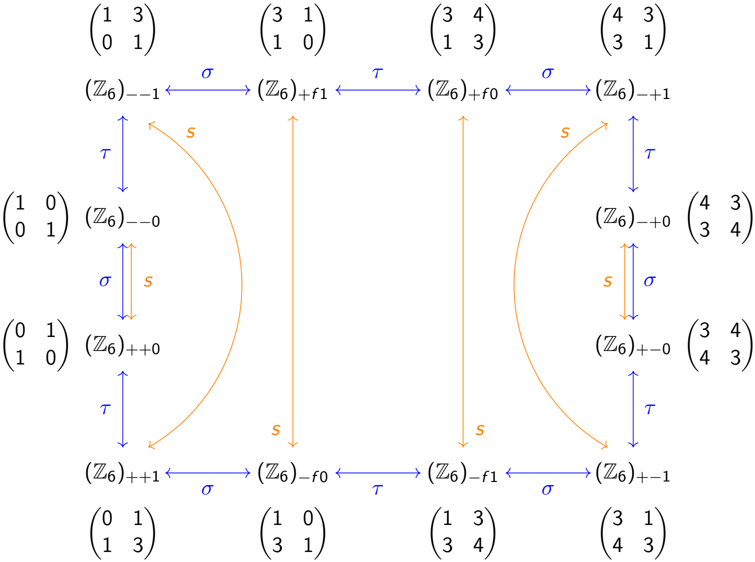
$$
\begin{array}{llllllll}
L_1 = \{(0,0), & (0,1), & (0,2), & (0,3), & (0,4), & (0,5)\} & \to & (\mathbb{Z}_6)_{++} = \mathbb{Z}_3 \times \mathbb{Z}_2 \\
L_2 = \{(0,0), & (2,0), & (4,0), & (0,3), & (2,3), & (4,3)\} & \to & (\mathbb{Z}_6)_{-+} = \widehat{\mathbb{Z}}_3 \times \mathbb{Z}_2 \\
L_3 = \{(0,0), & (0,2), & (0,4), & (3,0), & (3,2), & (3,4)\} & \to & (\mathbb{Z}_6)_{+-} = \mathbb{Z}_3 \times \widehat{\mathbb{Z}}_2 \\
L_4 = \{(0,0), & (1,0), & (2,0), & (3,0), & (4,0), & (5,0)\} & \to & (\mathbb{Z}_6)_{--} = \widehat{\mathbb{Z}}_3 \times \widehat{\mathbb{Z}}_2 \\
L_5 = \{(0,0), & (0,2), & (0,4), & (3,1), & (3,3), & (3,5)\} & \to & (\mathbb{Z}_6)_{+f} = \mathbb{Z}_3 \times \mathbb{Z}_2^f \\
L_6 = \{(0,0), & (2,0), & (4,0), & (1,3), & (3,3), & (5,3)\} & \to & (\mathbb{Z}_6)_{-f} = \widehat{\mathbb{Z}}_3 \times \mathbb{Z}_2^f
\end{array}
$$

- The automorphism group is $\mathrm{Aut}_{\mathbb{Z}_6}(Q) = \mathbb{Z}_2^2 \times S_3$ with

$$
\mathrm{O}_6(Q) = S_3 \times \mathbb{Z}_2 \,, \qquad \mathrm{Aut}(\mathbb{Z}_6) = \mathbb{Z}_2.
$$

- There are $|O_6(Q)| = 12$ global forms transforming between each other by the following two topological manipulations

$$
\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \tau = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} .
$$


$\begin{pmatrix}1&3\\0&1\end{pmatrix}$ $\begin{pmatrix}3&1\\1&0\end{pmatrix}$ $\begin{pmatrix}3&4\\1&3\end{pmatrix}$ $\begin{pmatrix}4&3\\3&1\end{pmatrix}$
$(\mathbb{Z}_6)_{--1}$ σ $(\mathbb{Z}_6)_{+f1}$ τ $(\mathbb{Z}_6)_{+f0}$ σ $(\mathbb{Z}_6)_{-+1}$
τ s s τ
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $(\mathbb{Z}_6)_{--0}$ $(\mathbb{Z}_6)_{-+0}$ $\begin{pmatrix}4&3\\3&4\end{pmatrix}$
σ s s σ
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $(\mathbb{Z}_6)_{++0}$ $(\mathbb{Z}_6)_{+-0}$ $\begin{pmatrix}3&4\\4&3\end{pmatrix}$
τ s s τ
$(\mathbb{Z}_6)_{++1}$ σ $(\mathbb{Z}_6)_{-f0}$ τ $(\mathbb{Z}_6)_{-f1}$ σ $(\mathbb{Z}_6)_{+-1}$
$\begin{pmatrix}0&1\\1&3\end{pmatrix}$ $\begin{pmatrix}1&0\\3&1\end{pmatrix}$ $\begin{pmatrix}1&3\\3&4\end{pmatrix}$ $\begin{pmatrix}3&1\\4&3\end{pmatrix}$

Global variants with general N

- Absolute theories of $T_N[\mathbb{P}^1 \times \mathbb{P}^1]$ are determined by the topological boundary conditions of the SymTFT.

- Global variants of $T_N[\mathbb{P}^1 \times \mathbb{P}^1]$ are one-to-one correspondence with the subgroup $O_N(Q) \subset \mathrm{Aut}_{\mathbb{Z}_N}(Q)$. Generators of $O_N(Q)$ gives the possible topological manipulations among these global variants.

- It is convenient to associate each global variant with a dimensional 2 reps of $O_N(Q)$. That makes the action of the topological and dualities on these global variants apparent.

- Automorphism group up to $N = 20$:

N	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{Aut}_{\mathbb{Z}_N}(Q)$	S_3	$\mathbb{Z}_2^2$	$D_8 \times \mathbb{Z}_2$	D_8	$S_3 \times \mathbb{Z}_2^2$	D_{12}	$D_8 \times \mathbb{Z}_2^2$	D_{12}	$S_3 \times D_8$	D_{20}	$D_8 \times \mathbb{Z}_2^3$

13	14	15	16	17	18	19	20
D_{24}	$S_3 \times D_{12}$	$D_8 \times \mathbb{Z}_2^2$	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2^2$	D_{34}	$S_3 \times D_{12}$	D_{36}	$D_8 \times D_8 \times \mathbb{Z}_2$

Duality defect

- $T_N[\mathbb{P}^1 \times \mathbb{P}^1]$ is a CFT with central charge

$$c_L = 8N^3 - 2N - 6, \qquad c_R = 8N^3 - 4N - 4$$

- Conformal manifolds with moduli [Dedushenko,Gukov,Putrov, 17]...

$$R = \frac{\mathrm{Vol}(\mathbb{P}^1_A)}{\mathrm{Vol}(\mathbb{P}^1_B)}$$

- The duality s changes the coupling constant into

$$R \xrightarrow{s} R^{-1} \ .$$

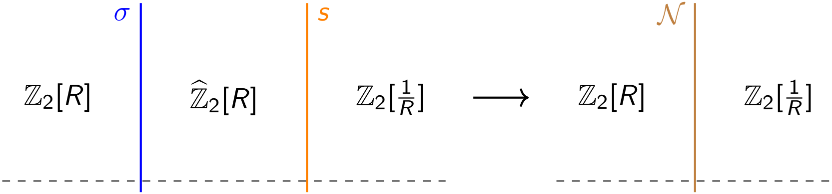
The self-dual coupling under s is $R = 1$.

- Find some topological manipulation $G(\sigma, \tau)$ which can undo the action of s and map the global variant to itself, i.e.

$$s^t M G = M,$$

with $G(\sigma, \tau) \in \mathrm{Aut}_{\mathbb{Z}_N}(Q)$.

- For example, at $R = 1$, $T_2[\mathbb{P}^1 \times \mathbb{P}^1]$ admits a duality defect $N = \sigma s$.



- Duality defects of $T_N[\mathbb{P}^1 \times \mathbb{P}^1]$ at $R = 1$ are:

N	Theory	Defects
2	$(\mathbb{Z}_2)_m, (\widehat{\mathbb{Z}}_2)_m$	$\tau^m \sigma s \tau^m$
2	$(\mathbb{Z}_2^f)_m$	$\tau^m \tau s \tau^m$
p	$\mathbb{Z}_p, \widehat{\mathbb{Z}}_p$	σs
4	$(\mathbb{Z}_4)_m, (\widehat{\mathbb{Z}}_4)_m, (\mathbb{Z}_4^f)_m, (\widehat{\mathbb{Z}}_4^f)_m$	$\tau^m \sigma s \tau^m$
6	$(\mathbb{Z}_6)_{\pm\pm m}$	$\tau^m \sigma s \tau^m$
6	$(\mathbb{Z}_6)_{\pm f m}$	$\sigma \tau \sigma \tau \sigma s$

- Tambara-Yamagami fusion categories $TY(\mathbb{Z}_N)$ with fusion rule

$$\eta^N = 1, \quad \eta \times \mathcal{N} = \mathcal{N}, \quad \mathcal{N} \times \mathcal{N} = \sum_{i=0}^{N-1} \eta^i,$$

where η is a $\mathbb{Z}_N$ line.

Connected sum of $\mathbb{P}^1 \times \mathbb{P}^1$

- Intersection form is

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

- SymTFT: $\mathbb{Z}_N \times \mathbb{Z}_N$ gauge theory

$$S_{3d} \;=\; \frac{N}{2\pi} \int_{W_3} a_1 \cup \delta \widehat{a}_1 + a_2 \cup \delta \widehat{a}_2 \; ,$$

where $a_i = \int_{b_i} c$ and $\hat{a}_i = \int_{f_i} c$ are $\mathbb{Z}_N$ cocycles on W_3.

- Discriminant group

$$\mathcal{D} = \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$$

Duality of $T_N[\#^2(\mathbb{P}^1 \times \mathbb{P}^1)]$

- Mapping class group of $\mathrm{MCG}(\#^2(\mathbb{P}^1 \times \mathbb{P}^1))$ is generated by

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

- We introduce three geometric parameters

$$R_1 = \frac{x}{y} = \frac{V_{f_1}}{V_{b_1}}, \; R_2 = \frac{z}{w} = \frac{V_{f_2}}{V_{b_2}}, \; R_3 = \frac{y}{z} = \frac{V_{b_1}}{V_{f_2}}.$$

- Action of on the parameters:

$$S \cdot R_1 = R_1 \ , \ S \cdot R_2 = \frac{1}{R_2} \ , \ S \cdot R_3 = R_2 R_3$$
$$D \cdot R_1 = R_2 \ , \ D \cdot R_2 = R_1 \ , \ D \cdot R_3 = \frac{1}{R_1 R_2 R_3}$$
$$T \cdot R_1 = \frac{R_1 R_2 R_3}{R_2 R_3 - 1} \ , \ T \cdot R_2 = R_2 + R_1 R_2 R_3 \ , \ T \cdot R_3 = \frac{1 - R_2 R_3}{R_1 R_2 R_3 + R_2}$$
$$W \cdot R_1 = R_1 \ , \ W \cdot R_2 = R_2 \ , \ W \cdot R_3 = -R_3 \, .$$

- Fixed points are extended loci in the conformal manifold.

$$S: \ (R_1, 1, R_3), \quad D: \ (R_1, \frac{1}{R_1}, \pm 1)$$
$$T: \ (0, R_2, \frac{1}{2R_2}), \quad W: \ (R_1, R_2, 0)$$

Global variants of $N = 3$

- Topological boundary conditions:

$$L_1 = \{(0,0,0,0),(0,0,0,1),(0,1,0,0),(0,1,0,1),(0,0,0,2),(0,1,0,2),(0,2,0,0),(0,2,0,1),(0,2,0,2)\}$$
$$L_2 = \{(0,0,0,0),(0,0,0,1),(1,0,0,0),(1,0,0,1),(0,0,0,2),(1,0,0,2),(2,0,0,0),(2,0,0,1),(2,0,0,2)\}$$
$$L_3 = \{(0,0,0,0),(0,0,1,0),(0,1,0,0),(0,1,1,0),(0,0,2,0),(0,1,2,0),(0,2,0,0),(0,2,1,0),(0,2,2,0)\}$$
$$L_4 = \{(0,0,0,0),(0,0,1,0),(1,0,0,0),(1,0,1,0),(0,0,2,0),(1,0,2,0),(2,0,0,0),(2,0,1,0),(2,0,2,0)\}$$
$$L_5 = \{(0,0,0,0),(0,1,0,1),(0,2,0,2),(1,0,2,0),(1,1,2,1),(1,2,2,2),(2,0,1,0),(2,1,1,1),(2,2,1,2)\}$$
$$L_6 = \{(0,0,0,0),(1,0,1,0),(0,1,0,2),(0,2,0,1),(1,1,1,2),(1,2,1,1),(2,0,2,0),(2,1,2,2),(2,2,2,1)\}$$
$$L_7 = \{(0,0,0,0),(0,1,1,0),(0,2,2,0),(1,0,0,2),(1,1,1,2),(1,2,2,2),(2,0,0,1),(2,1,1,1),(2,2,2,1)\}$$
$$L_8 = \{(0,0,0,0),(1,0,0,1),(0,1,2,0),(0,2,1,0),(1,1,2,1),(1,2,1,1),(2,0,0,2),(2,1,2,2),(2,2,1,2)\}$$

  There are 8 absolute theories with $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry.
- Automorphism group has order $|\mathrm{Aut}_{\mathbb{Z}_3}(\mathbb{Z}_3 \times \mathbb{Z}_3)| = 1152$. Since $|GL(3, \mathbb{Z}_3)| = 48$, one has that $|\mathcal{O}_3(Q)| = 24$.
- Topological manipulations: [Gaiotto Kulp 20']...
  - Stacking bosonic SPT: $v_2 \in H^2(\mathbb{Z}_3 \times \mathbb{Z}_3, U(1)) = \mathbb{Z}_3$
  - Gauging $\mathbb{Z}_3 \times \mathbb{Z}_3$ with possible SPT phases.
  - Gauging $\mathbb{Z}_3$ subgroup with generator embedded in the following as $(1,0)$, $(0,1)$, $(1,1)$, $(1,2)$

Duality defect

- Choose the 4-dimensional rep of each global variants in $\mathcal{O}_3(Q)$. For example

$$M_{L_1}^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

- Consider the finite subgroup generated by S and D that is $D_8 \subset \mathrm{MCG}(\#^2(\mathbb{P}^1 \times \mathbb{P}^1))$.

$$(DS)^t M_{L_1}^{(2)} \sigma_2 \sigma_3 = M_{L_1}^{(2)}$$

- At the self-dual coupling

$$(R_1, R_2, R_3) = (1, 1, \pm 1)\,.$$

The duality defects is $\mathcal{N} = \sigma_2 \sigma_3 SD$ described by $TY(D_8)$.

Hirzebruch surface

- Hirzebruch surface $\mathbb{F}_l$ with intersection form

$$Q = \begin{pmatrix} f\cdot f & f\cdot b \\ b\cdot f & b\cdot b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -l \end{pmatrix}.$$

- Twisted $\mathbb{Z}_N$ gauge theory [Dijkgraaf-Witten,90']...

$$S_{3d} = \frac{N}{2\pi}\int \hat{a}\wedge da - \frac{Nl}{4\pi}\int a\wedge da$$

  where $a = \int_b c$ and $\hat{a} = \int_f c$. The coefficients of the Dijkgraaf-Witten twist is integers in $\mathbb{Z}_{2N}$. Sufficient to consider $\mathbb{F}_1$.

- Mapping class group $\mathrm{MCG}(\mathbb{F}_1) = \mathbb{Z}_2^2$ with nontrivial $\mathbb{Z}_2$ given by

$$r = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

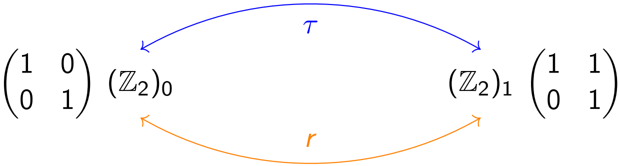
Global variants of $N = 2$

- Topological boundary condition:

$$L = \{(0,0), \quad (1,0)\} \to \mathbb{Z}_2$$

  There is only one absolute theory with anomalous $\mathbb{Z}_2$.
- Automorphism group $\mathrm{Aut}_{\mathbb{Z}_2}(Q) = \mathbb{Z}_2$, there are two global variants related by stacking Arf invariants. The rep in $GL(2, \mathbb{Z}_2)$ is

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$



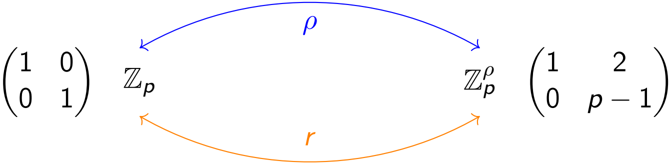
Global variants of $N = p$

- Topological boundary conditions:

$$L_1 = \langle (1,0) \rangle \;\to\; \mathbb{Z}_p, \qquad\qquad L_2 = \langle (1,2) \rangle \;\to\; \mathbb{Z}_p^{\rho}$$

- Automorphism group $\mathrm{Aut}_{\mathbb{Z}_p}(Q)$ is still $D_{2(p-1)}$. Taking into account the $\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^{\times}$, one has $\mathcal{O}_p(Q) = \mathbb{Z}_2$ with generator

$$\rho = \begin{pmatrix} 1 & 0 \\ 2 & p-1 \end{pmatrix}$$



Conclusion

Conclusion

- Using SymTFT, we study the global variants of the 2d theories $T_N[M_4]$ arsing from the compactification of 6d $\mathcal{N} = (2, 0)$ SCFTs of type A_{N-1} on 4-manifolds including $\mathbb{P}^1 \times \mathbb{P}^1$, connected sums of $\mathbb{P}^1 \times \mathbb{P}^1$ and Hirzebruch surfaces.
- The global variants transform between each other by the topological manipulations and dualities. We identify the topological manipulations with automorphism of SymTFTs and dualities as the mapping class groups.
- From the web of global variants, at the self-dual point in the conformal manifold, we are able to construct duality defects in some of $T_N[M_4]$.

Thank you!