3d N=2 from M-theory on CY4 and IIB brane box Part II

McKay correspondence and brane box

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Overview

1. Super quick recap
2. McKay correspondence and SCFT data
3. Brane box construction

Super quick recap

- ... Done (Thank you Yi-Nan, and the organizers!)
- Are there other ways to engineer 3D $\mathcal{N} = 2$?
  - Geometry: $\mathbb{C}^4$ orbifolds
  - Brany: Brane box

McKay correspondence: Why?

- A rich class of varieties in geometric engineering
- Classified (2D, 3D, 4D, those we do care about) ☺
- Can be studied algebraically
  - Conjugacy classes: divisors
  - Irreps: intersection pairing (?)
  - Hypersurface/CICY equations: Most math/phys statements can be concretely checked

McKay correspondence: 2D

Familiar

- Physics: M-theory on $\mathbb{C}^2/\Gamma$, $\Gamma \subset SU(2)$

- Math:
    - Classification

$$\mathbb{Z}_n\text{(C)yclic, } \mathbb{D}_n\text{ihedral, } \mathbb{T}\text{etrahedral, } \mathbb{O}\text{ctahedral, } \mathbb{I}\text{cosahedral}$$

    - Crepant resolution? Yes
    - Conjugacy classes of Γ

$$\text{Conjugacy classes} \leftrightarrow \text{Divisors}$$

    - Irreps

$$\text{Adjacency matrix} \leftrightarrow \text{Intersection pairing}$$

    - Equations: e.g. $\mathbb{Z}_n : \{xy = z^n\} \subset \mathbb{C}^3$

McKay correspondence: 3D

Less familiar (if so please read [Acharya-Lambert-Najjar-Svanes-JT] [JT-Wang])

- Physics: M-theory on $X_3 = \mathbb{C}^3/\Gamma$, $\Gamma \subset SU(3)$
- Math:
  - Classification: 107th anniversary of [Blichfeldt]
  - Crepant resolution? Yes [Ito-Reid], [Ito-Nakajima]
  - Conjugacy classes: Yes they are divisors but what divisor?
  - Irreps: "Adjacency matrix" $\leftrightarrow$ Intersection pairing
  - Equations: e.g. $T_n : \{xyz = w^n\} \subset \mathbb{C}^4$

McKay correspondence: 3D

[Acharya-Lambert-Najjar-Svanes-JT], [JT-Wang]

- Conjugacy classes

Definition 1 (Ito)

$\mathrm{age}(\mathfrak{g}) = \frac{1}{r}(a+b+c)$ for $\mathfrak{g} = \mathrm{diag}\left(e^{2\pi ia/r}, e^{2\pi ib/r}, e^{2\pi ic/r}\right)$, $|\Gamma_m|$: $\#$ of $\mathrm{age}(\mathfrak{g}) = m$.

Theorem (Ito-Reid)

$b_{2m}(\widetilde{X_3}) = |\Gamma_m|$ *(Existence of $\widetilde{X_3}$ is proved by Ito-Reid)*

Theorem (Ito-Reid translated in Acharya-Lambert-Najjar-Svanes-JT & JT-Wang)

$|\Gamma_1| = \#$ *of (exceptional) prime divisors* $= r + f$, $|\Gamma_2| = \#$ *of compact divisors* $= r$

McKay correspondence: 3D

- Irreps

Definition 2 (Ito-Nakajima)

$$Q \otimes \rho_k = \oplus_l a^{(1)}_{kl} \rho_l, \ (Q \wedge Q) \otimes \rho_k = \oplus_l a^{(2)}_{kl} \rho_l \ (Q \text{ is natural 3D rep})$$

Theorem (Ito-Nakajima)

The intersection pairing on $K^c(\widetilde{X_3})$ is $(S_k^\vee, S_l) = a^{(2)}_{kl} - a^{(1)}_{kl} = a^{(1)}_{lk} - a^{(1)}_{kl}$

Theorem (Ito-Nakajima translated in JT-Wang)

The SNF of the pairing between $H_*(X_3, \partial X_3)$ and $H_{6-*}(X_3, \partial X_3)$ = SNF$(a^{(1)}_{lk} - a^{(1)}_{kl})$

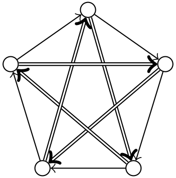
$$\text{SNF}(a^{(1)}_{lk} - a^{(1)}_{kl}) \to \text{1-form symmetry}$$ [Morrison-Schäfer-Nameki-Willett]

McKay correspondence: 3D

- An example: $\mathbb{Z}_5 = \text{diag}\left(e^{\frac{2\pi 2 i}{5}}, e^{\frac{2\pi 2 i}{5}}, e^{\frac{2\pi 1 i}{5}}\right)$

- Conjugacy classes: $r = |\Gamma_2| = 2,\ f = |\Gamma_1| - |\Gamma_2| = 0$

- McKay quiver:



$$\text{SNF}(a^{(1)}_{kl} - a^{(1)}_{lk}) = \text{diag}(5,5,1,1,0) \Rightarrow \mathbb{Z}_5 \text{ 1-form symmetry} \qquad (1)$$

McKay correspondence: 4D

Not familiar and not clear, both math and phys

- Physics: M-theory on $X_4 = \mathbb{C}^4/\Gamma$, $\Gamma \subset SU(4)$
- Math:
    - Classification: Done in [Hanany-He]
    - Crepant resolution? Open problem 1: Not known when Γ is non-abelian
    - Conjugacy classes: Yes they are divisors
    - Irreps: "Some matrix" $\leftrightarrow$?

McKay correspondence: 4D

- Conjugacy classes

Theorem (Joyce)

$$b_{2k}(\widetilde{X}) = |\Gamma_k|, \textit{ if } \pi : \widetilde{X_n} \to X_n \textit{ exists}$$

Corollary (Joyce translated in Najjar-JT-Wang)

$$|\Gamma_1| = r + f,\ |\Gamma_{n-1}| = r$$

McKay correspondence: 4D

- Irreps

Definition 3 (Ito-Nakajima)

$$\left(\bigwedge^i Q\right) \otimes \rho_k = \oplus_l a_{kl}^{(i)} \rho_l, \ (Q \text{ is natural } n\text{D rep})$$

Definition 4 (Ito)

$$\mathcal{A} = \sum_i (-1)^i a_{kl}^{(i)}$$

Open problem 2: What does $\mathcal{A}$ correspond to in $\widetilde{X_4}$?

- Short answer: we don't know.
- Longer answer: we know what it does NOT correspond to – NOT intersection pairing

⇒ Open problem 3: How to read intersection pairing off McKay quiver?

McKay correspondence: 4D

What do we know?

- Conjugacy classes (assuming $\exists \widetilde{X_4}$)

Group	Generators	Order	χ	$\lvert\Gamma_1\rvert$	$\lvert\Gamma_2\rvert$	$\lvert\Gamma_3\rvert$
Primitive simple groups						
I	F_1, F_2, F_3	240	18	3	13	1
II	F_1, F_2', F_3'	60	5	2	2	0
III	F_1, F_2, F_3, F_4	1440	26	4	19	2
IV	S, T, W	5040	16	2	13	0
V	S, T, R	336	11	2	8	0
VI	T, C, D, V, F	51840	34	8	24	1
Groups having simple normal primitive subgroups						
VII	F_1, F_2, F_3, F''	480	24	5	17	1
VIII	F_1, F_2', F_3', F'	480	28	5	20	2
IX	F_1, F_2, F_3, F_4, F''	2880	34	7	24	2

McKay correspondence: 4D

- Hypersurface equation

X_4 is (sometimes) a hypersurface in $\mathbb{C}^5$, e.g.:

$$
\begin{aligned}
F_{\mathrm{XXXI}}(u_1,u_2,u_3,u_4,u_5) = {} & u_1^{12} - 15u_1^{10}u_2 + 12u_1^9u_3 + 90u_1^8u_2^2 - 144u_1^7u_2u_3 - 246u_1^6u_2^3 \\
& + 60u_1^6u_3^2 + 576u_1^5u_2^2u_3 + 261u_1^4u_2^4 - 468u_1^4u_2u_3^2 - 144u_1^4u_2u_4 \\
& - 768u_1^3u_2^3u_3 + 128u_1^3u_3^3 + 144u_1^3u_3u_4 - 27u_1^2u_2^5 + 900u_1^2u_2^2u_3^2 \\
& + 432u_1^2u_2^2u_4 + u_5^{2n}\left(4320u_1^4 - 18144u_1^2u_2 + 13824u_1u_3 + 7776u_2^2\right) \\
& + u_5^n\left(-108u_1^8 + 972u_1^6u_2 - 720u_1^5u_3 - 2988u_1^4u_2^2 + 4032u_1^3u_2u_3\right. \\
& +2916u_1^2u_2^3 - 1440u_1^2u_3^2 + 864u_1^2u_4 - 4464u_1u_2^2u_3 - 216u_2^4 + 2016u_2u_3^2 \\
& \left.+1728u_2u_4\right) + 36u_1u_2^4u_3 - 480u_1u_2u_3^3 - 720u_1u_2u_3u_4 - 12u_2^3u_3^2 \\
& + 96u_3^4 + 288u_3^2u_4 + 864u_4^2 - 55296u_5^{3n}.
\end{aligned}
$$

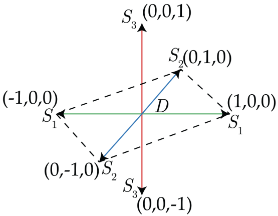
↑ A really good test case for non-abelian Γ. (Had I got a better Mac ☹)

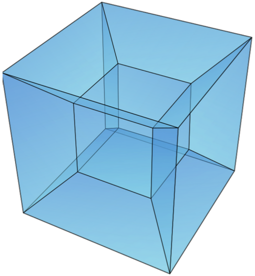
McKay correspondence: Some comments

- No crepant resolution?
  Some discussion in [Arras-Grassi-Weigand] but Joyce's theorem no longer holds
- Deformations?
  Open problem 4: Most singular loci are non-isolated ☹
- Finer structure?
  Triple (quadruple) intersections can NOT be read off McKay correspondence
- CY5?
  We are aware of no classification

Brane box: Why?

- Brane box is dual to toric geometry [Leung-Vafa]





Brane box: Why?

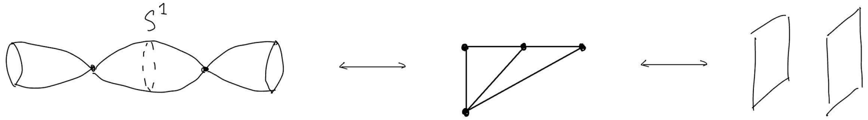
- DOF of $\mathcal{T}_{3D} \leftrightarrow$ Brane movements
  - CB parameters
  - ECB parameters
- Dynamics of $\mathcal{T}_{3D} \leftrightarrow$ Diagrammatic of brane box
- States of $\mathcal{T}_{3D} \leftrightarrow$ Stringy/brany states of the brane box
- HW moves and more

Brane box: Construction and duality with toric geometry

- Warm-up 1: 7D

	0	1	2	3	4	5	6	7	S^1_8	S^1_9	S^1_{10}
$\mathrm{KK7M}^{(\#)}$	✓	✓	✓	•	•	•	✓	✓	✓	✓	TN

KK7M $\leftrightarrow$ toric CY2 $\leftrightarrow$ parallel D6's (along 5)

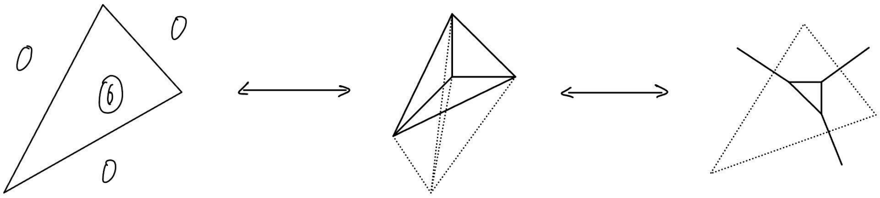


Brane box: Construction and duality with toric geometry

- Warm-up 2: 5D

	0	1	2	3	4	5	6	7	S^1_8	S^1_9	S^1_{10}
KK7M$^{(\#)}$	✓	✓	✓	•	•	•	✓	✓	✓	✓	TN
KK7M$^{(9)}$	✓	✓	✓	•	•	✓	•	✓	✓	TN	✓

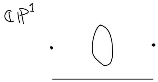
KK7M $\leftrightarrow$ toric CY3 $\leftrightarrow$ (p, q) 5-brane web (in 56)

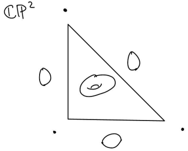


Brane box: Construction and duality with toric geometry

What do we learn from the warm-ups?

- Toric geometry: various degenerations of T^n fiber





- KK7M: various degenerations of S^1's
- Branes: KK7M via M-theory/type II duality

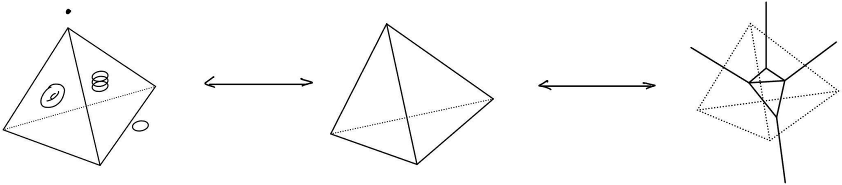
$\Rightarrow$ Edge of toric diagram $\leftrightarrow$ Plane/Brane

Brane box: Construction and duality with toric geometry

- Main case: 3D

	0	1	2	3	4	5	6	7	S^1_8	S^1_9	S^1_{10}
$\mathrm{KK7M}^{(\#)}$	✓	✓	✓	●	●	●	✓	✓	✓	✓	TN
$\mathrm{KK7M}^{(9)}$	✓	✓	✓	●	●	✓	●	✓	✓	TN	✓
$\mathrm{KK7M}^{(8)}$	✓	✓	✓	●	●	✓	✓	●	TN	✓	✓

KK7M $\leftrightarrow$ toric CY4 $\leftrightarrow$ Some intersecting planes (in 567)



Brane box: planes as branes

- What are those “planes”?
  - 7D: KK7M $\xrightarrow{\text{M/IIA}}$ D6
  - 5D: KK7M $\xrightarrow{\text{M/IIB}}$ (p, q) 5-brane

S^1-reduction

	0	1	2	3	4	5	6	7	S^1_8	S^1_9	S^1_{10}
$\mathrm{KK7M}^{(\#)}$	✓	✓	✓	•	•	•	✓	✓	✓	✓	TN
$\mathrm{KK7M}^{(9)}$	✓	✓	✓	•	•	✓	•	✓	✓	TN	✓
$\mathrm{KK7M}^{(8)}$	✓	✓	✓	•	•	✓	✓	•	TN	✓	✓

T-dual

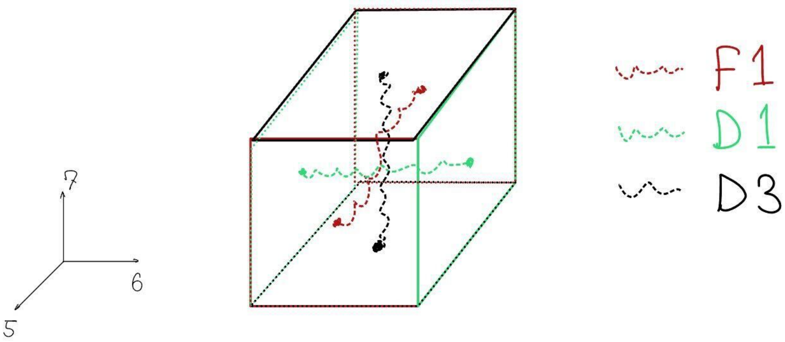
- 3D: KK7M $\xrightarrow{\text{M/IIB}}$ (D5, NS5, KK6B) on T^2

Brane box: planes as branes

- Planes/branes labeled by its normal direction (p, q, r) in 567
- Strings/branes labeled by the branes they end on in 567

	0	1	2	3	4	5	6	7	S^1_8	$\widetilde{S}^1_9$
$D_5^{(1,0,0)}$	✓	✓	✓	•	•	•	✓	✓	✓	•
$NS_5^{(0,1,0)}$	✓	✓	✓	•	•	✓	•	✓	✓	•
$KK6B^{(0,0,1)}$	✓	✓	✓	•	•	✓	✓	•	TN	✓
$F_1^{(1,0,0)}$	✓	•	•	•	•	✓	•	•	•	•
$D_1^{(0,1,0)}$	✓	•	•	•	•	•	✓	•	•	•
$D_3^{(0,0,1)}$	✓	•	•	•	•	•	•	✓	✓	✓

Brane box: planes as branes



Puzzle: where is the $SL(3, \mathbb{Z})$?

Brane box: planes as branes

T^2

0	1	2	3	4	5	6	7	S^1_8	$\widetilde{S}^1_9$
✓	✓	✓				✓	✓	✓	

10D algebra:

$$\{Q^\alpha, Q^\beta\}^{\mathbb{R}^{1,7}\times\widetilde{S}^1_9\times S^1_8} \supset (\Gamma^{M_1\cdots M_4 8}C^{-1})\, Z^{(p,q)}_{M_1\cdots M_4 8} + (\Gamma^{M_1\cdots M_4 9}C^{-1})\; Z^{(r)}_{M_1\cdots M_4 98}\, k^8$$

8D algebra:

$$\{Q^\alpha, Q^\beta\}^{8\mathrm{D}} \supset (\Gamma^{M_1\cdots M_4}C^{-1})\; Z^{(p,q,r)}_{M_1\cdots M_4}$$

$\Rightarrow$ $SL(3,\mathbb{Z})$ “central charge” Z

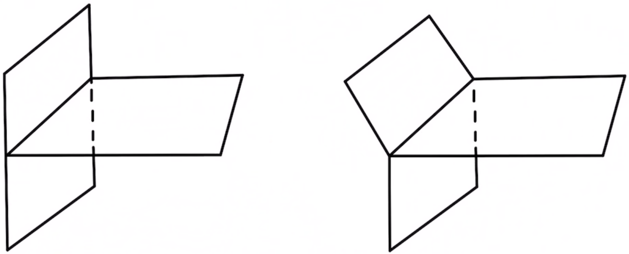
Brane box: planes as branes

p, *q*, *r* labels are symmetric in 8D.

	0	1	2	3	4	5	6	7
(1,0,0) 4-brane	✓	✓	✓	●	●	●	✓	✓
(0,1,0) 4-brane	✓	✓	✓	●	●	✓	●	✓
(0,0,1) 4-brane	✓	✓	✓	●	●	✓	✓	●
(1,0,0)-string	✓	●	●	●	●	✓	●	●
(0,1,0)-string	✓	●	●	●	●	●	✓	●
(0,0,1)-string	✓	●	●	●	●	●	●	✓

Brane box: planes as branes

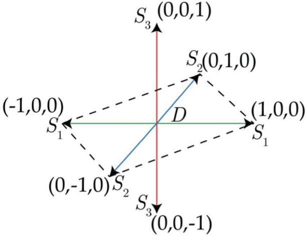
- Some features (c.f. 5-brane web):
  - Static condition for branes: $\sum \tau_{(p,q,r)} = 0 \Rightarrow \sum p = \sum q = \sum r = 0$
  - Charge conservation for strings: $\sum \tau_{(p,q,r)} = 0 \Rightarrow \sum p = \sum q = \sum r = 0$
  - Brane bending: e.g. $(0,1,0)$ bends $(0,0,1)$ to $(0,1,1)$

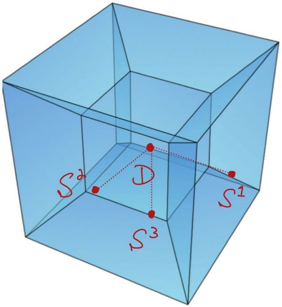


Brane box: planes as branes

(p,q,r) 4-brane box	toric CY4
brane box	moment map
(p,q,r) 4-brane	locus of degenerating (p,q,r)-cycle of T^3
finite cell	compact divisor
semi-infinite cell	non-compact divisor
semi-infinite 4-brane	normal direction to compact divisor
Charge conservation	Calabi-Yau condition

Brane box: planes as branes





$$\text{CY} \Leftrightarrow N_D = K_D \Leftrightarrow T^n \text{ actions match} \Leftrightarrow \sum_C (p,q,r) = \sum_{NC} (p,q,r)$$

Brane box: planes as branes

- Bonus: M5-brane realization

$$(\text{M5,M5,M5}) \xrightarrow{S^1_{10}} (\text{D4,NS5,NS5}) \xrightarrow{S^1_8} (D5, NS5, KK6B) \text{ on } T^2$$

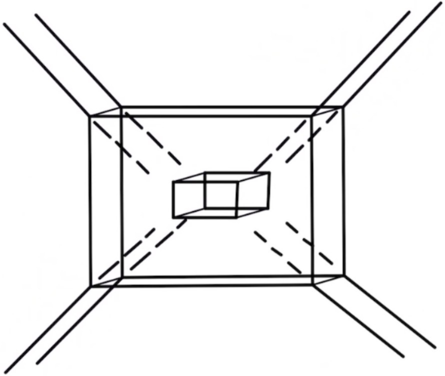
	0	1	2	3	4	5	6	7	S^1_8	S^1_9	S^1_{10}
$\text{M5}^{(1,0,0)}$	✓	✓	✓	•	•	•	✓	✓	•	•	✓
$\text{M5}^{(0,1,0)}$	✓	✓	✓	•	•	✓	•	✓	✓	•	•
$\text{M5}^{(0,0,1)}$	✓	✓	✓	•	•	✓	✓	•	•	✓	•

Feat. Junya Yagi: slightly different but maybe related to the construction in [Yagi '22]

Brane box: CB physics

- Test 1: rank(CB)

$$\text{rank(CB)} = \#\ \text{local deformations}$$



Brane box: CB physics

- Each (p, q, r) 4-brane $\rightarrow$ 1 DOF along (p, q, r)
- Each (internal) edge $\rightarrow$ 1 constraint
- Each vertex $\rightarrow$ 1 redundancy
- Connectivity $\rightarrow$ 1 constraint

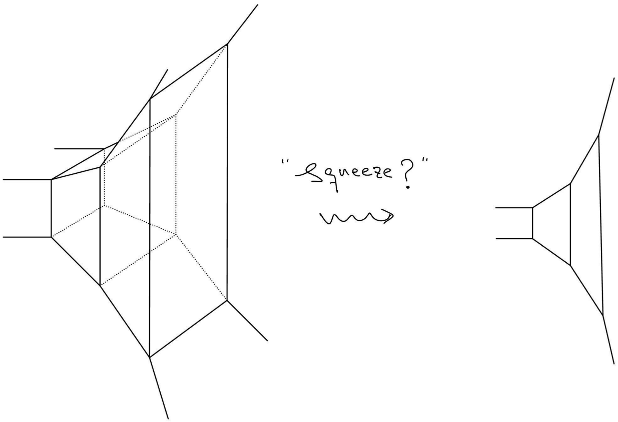
$$\Rightarrow \text{rank(CB)} = \#\ \text{local Def} = \#\ \text{F} - \#\ \text{E} + \#\ \text{V} - 1$$

Mighty Euler: $\#\ \text{V} - \#\ \text{E} + \#\ \text{F} - \#\ \text{C} = 1$

$\text{rank(CB)} = \#\ \text{C}$ (in Yi-Nan's talk)

Brane box: CB physics

- Example: Feat. Sung-Soo Kim & Futoshi Yagi



Brane box: CB physics

- $U(1)$ coupling in UV

$$S \supset \frac{1}{g^2}\int d^3x \, \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \Rightarrow \frac{1}{g^2} = g_{11}^{CB}$$

Kinetic energy $\Leftarrow$ "Breathing" of compact cell $\Rightarrow$ From all branes

$$\delta\phi \to \delta KE = \sum_i \frac{1}{2}m_i(\delta_i\delta\phi)^2 \simeq \sum_i \frac{1}{2}A_i(\delta_i\delta\phi)^2$$

$\Rightarrow \frac{1}{g^2} = \sum_i \frac{1}{2}A_i(\delta_i)^2$ (c.f. 5-brane web, real scalar from one higher dim)

Brane box: CB physics

- $U(1)$ coupling in UV

(p, q, r) 4-brane box	toric CY4
brane box	moment map
finite cell	compact divisor
finite facet	"divisor of divisor"

$$\Rightarrow A_i \simeq \mathrm{Vol}(S_i) \text{ for } S_i \subset D$$

$$\frac{1}{2}\sum_i A_i(\delta_i)^2 \simeq \sum_i \mathrm{Vol}(S_i) = \mathrm{Vol}(-K_D) \text{ (in Yi-Nan's talk)}$$

Brane box: CB physics

- States
  - Mass

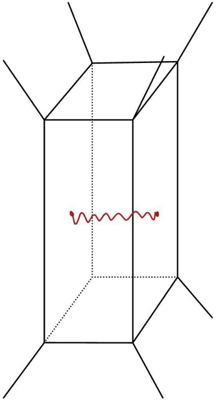
$$m(\text{particle}) = m(\text{string}) \propto l(\text{string})$$

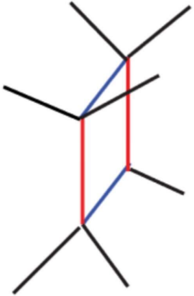
  - Charge

$$Q_e = -N_b := \# \text{ of finite 4-branes it ends on}$$

Brane box: CB physics

- Enhancements
  - Non-abelian: $m(\text{W-boson}) \to 0$

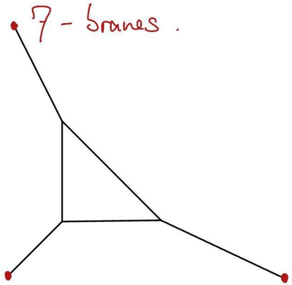


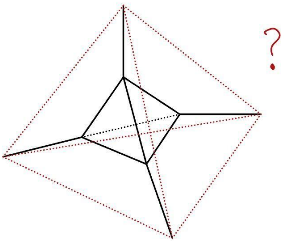


$$\frac{1}{g_0^2} \simeq \mathrm{Vol}(B) \simeq A_{(1,0,0)}$$

Brane box: Codim-2 branes and flavor

- Flavor? Need more structures





Brane box: Codim-2 branes and flavor

Requirements:

- Compatible i.e. supersymmetric
- 4-brane can end on them $\Leftrightarrow$ KK7M can end on them
- Strings/D3 can end on them $\Leftrightarrow$ M2-brane can end on them
- Carry certain charges in 8D IIB
- Exhibit certain $SL(3,\mathbb{Z})$ properties

Deus ex machina: Yes there are 5-branes in 8D

- Codim-2 in 8D (c.f. 7-branes in 10D)

Brane box: Codim-2 branes and flavor

- SUSY configuration

	0	1	2	3	4	5	6	7	S^1_8	S^1_9	S^1_{10}
KK7M	✓	✓	✓	✓	✓	•	•	✓	✓	•	TN
KK7M	✓	✓	✓	✓	✓	•	•	✓	✓	TN	•
KK7M	✓	✓	✓	✓	✓	✓	•	•	•	TN	✓
KK7M	✓	✓	✓	✓	✓	✓	•	•	TN	•	✓
KK7M	✓	✓	✓	✓	✓	•	✓	•	•	✓	TN
KK7M	✓	✓	✓	✓	✓	•	✓	•	TN	✓	•

S^1_{10} → D6, KK6A → (S^1_9) D7, NS7

→ KK6A, KK6A → NS5, 5^2_2

→ D6, KK7A → D5, 5^2_3

Exotic branes.

$$\text{KK7M} \xrightarrow{\text{Reduce on } S^1_{10},\ \text{T-dual on } S^1_9} \text{Exotic IIB branes} \xrightarrow{\text{On } T^2} \text{8D 5-branes}$$

Compatible i.e. supersymmetric ✓

[de Boer-Shigemori]

Brane box: Codim-2 branes and flavor

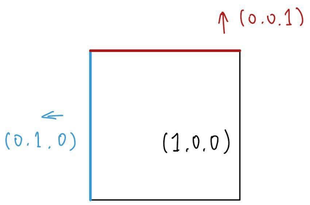
- Charges can be read off TN

$$(1,0,0)\text{ 5-brane: } (\widetilde{S}^1_9 \times S^1_8\text{-wrapped D7, D5}),$$
$$(0,1,0)\text{ 5-brane: } (\widetilde{S}^1_9 \times S^1_8\text{-wrapped NS7, NS5}),$$
$$(0,0,1)\text{ 5-brane: } (5^2_3, 5^2_2).$$

$\Rightarrow$ determines which 4-brane ends on which 5-brane... but why two?

Brane box: Codim-2 branes and flavor

- Kindergarten math



5-brane labeled by not only charge, also direction

Facet: 4-brane. Edge: 5-brane.

Brane box: Codim-2 branes and flavor

- Charges and directions

$$
\begin{aligned}
(\widetilde{S}^1_9 \times S^1_8\text{-wrapped D7, D5}) &\to ((1,0,0)_{(0,1,0)}, (1,0,0)_{(0,0,1)}) \\
(\widetilde{S}^1_9 \times S^1_8\text{-wrapped NS7, NS5}) &\to ((0,1,0)_{(1,0,0)}, (0,1,0)_{(0,0,1)}), \\
(5^2_3, 5^2_2) &\to ((0,0,1)_{(1,0,0)}, (0,0,1)_{(0,1,0)}).
\end{aligned}
$$

4-brane can end on them $\Leftrightarrow$ KK7M can end on them $\checkmark$

Brane box: Codim-2 branes and flavor

- Strings/branes end on them?

In IIB we have:

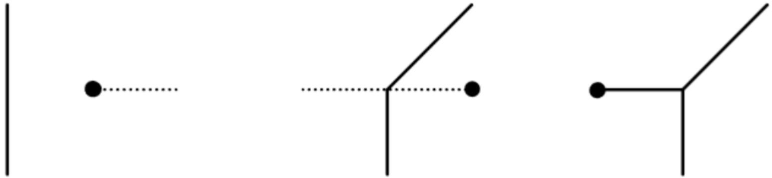
$$\begin{aligned} \text{F1 ends on} \quad & \widetilde{S}^1_9 \times S^1_8 - \text{wrapped D7 and D5}, \\ \text{D1 ends on} \quad & \widetilde{S}^1_9 \times S^1_8 - \text{wrapped NS7 and NS5}, \\ \widetilde{S}^1_9 \times S^1_8 - \text{wrapped D3 ends on} \quad & 5^2_2 \text{ and } 5^2_3. \end{aligned}$$

Reassures that (p, q, r)-string ends on (p, q, r) 5-brane ✓

See [Lu-Roy] for:
- Carry certain charges in 8D IIB ✓
- Exhibit certain $SL(3, \mathbb{Z})$ properties ✓

Brane box: Codim-2 branes and flavor

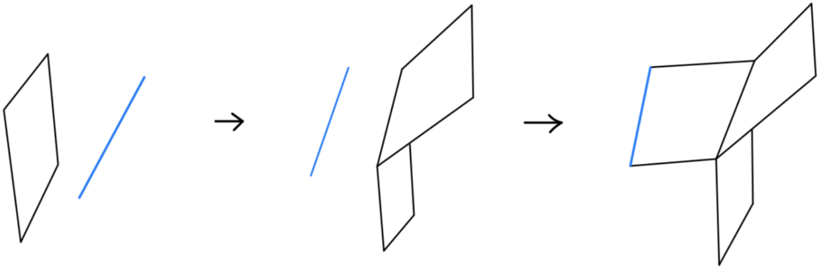
- Hanany-Witten moves in brane web



No branch cut → Branch cut → HW move

Brane box: Codim-2 branes and flavor

- Hanany-Witten moves in brane box



Brane box: Codim-2 branes and flavor

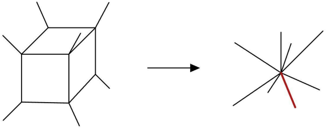
- Some comments:
  - U-duality manifest in KK7M or 8D
  - Many things resemble 5-brane web
  - Considerably more complicated due to various exotic branes

Brane box: Codim-2 branes and flavor

- Flavor rank

$$\text{rank}(F) = \#\text{ global deformations}$$

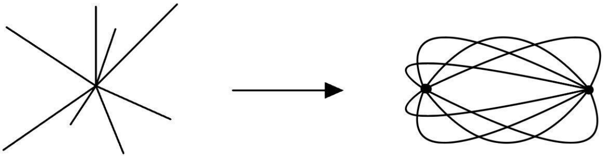
  - Each semi-infinite 4-brane → 1 DOF
  - Three overall translations are redundant
  - Each (external) edge → 1 constraint
  - One edge constraint is redundant



$$\text{rank}(F) = \#\text{ global deformations} = F_X - E_X - 2$$

Brane box: Codim-2 branes and flavor

- Flavor rank – Euler is our savior



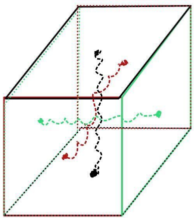
$$\mathbb{R}^3 \to S^3 \quad \text{and} \quad 2 - E_X + F_X - C_X = \chi(S^3) = 0$$

$$\Rightarrow \text{rank}(F) = C_X - 4, \text{ (in Yi-Nan's talk)}$$

Brane box: Codim-2 branes and flavor

- Charges (in example)

Measure (Q_e, Q_A, Q_B) of



$\frac{1}{2}N$

=

	$Q_{(1,0,0)}$	$Q_{(0,1,0)}$	$Q_{(0,0,1)}$
$T_{(1,0,0)}$	1	0	0
$T_{(0,1,0)}$	0	1	0
$T_{(0,0,1)}$	0	0	1

Brane box: Codim-2 branes and flavor

- Charges (in example)

$$
\begin{aligned}
Q_e &= -2\left(Q_{(1,0,0)} + Q_{(0,1,0)} + Q_{(0,0,1)}\right),\\
Q_A &= Q_{(1,0,0)} - Q_{(0,1,0)},\\
Q_B &= Q_{(0,1,0)} - Q_{(0,0,1)}.
\end{aligned}
$$

	Q_e	Q_A	Q_B
$T_{(1,0,0)}$	-2	1	0
$T_{(0,1,0)}$	-2	-1	1
$T_{(0,0,1)}$	-2	0	-1

This is a choice, in accordance with $F_1 = S_1 - S_2$, $F_2 = S_2 - S_3$ (in Yi-Nan's talk)

Brane box: Codim-2 branes and flavor

- Charges (in general)
  - No canonical assignment of charges
  - Has to be checked against the choice in geometry
  - Determined/under-determined system $\rightarrow$ A choice is generically guaranteed

Outlook

- Outlook
  - Flavor symmetry enhancement from brane box?
  - GTP (Generalized Toric Polygons, or toric-like diagram) from brane box?
  - Higgs branch from brane box?
  - Non-compact CY metric from TN's?

Thanks