On Toplogical Defect Lines in Para-Fermionic CFTs

Jin Chen (Xiamen University)

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The work is based on 2208.02757 with C.-M. Chang and F. Xu,

2309.01914 with B. Haghighat and Q.-R. Wang.

Outline:

* TDLs in 2D Parafermionic CFTs

* An 3D Anyon Condensation Approach

* Examples.

* Summary & Outlook

I. TDLs in 2D Parafermionic CFTs

— In a QFT, a continuous symmetry G ⟿

1-form conserved current J by Noether procedure.

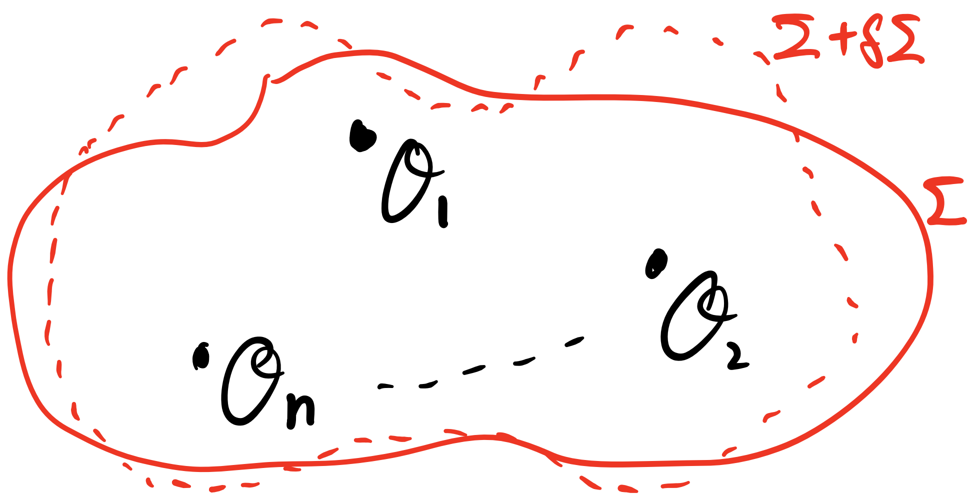
⇒ the charge $Q(\Sigma) = \int_{\Sigma} {*J}$. depends on a surface Σ.
(codimension 1)

I. TDLs in 2D Parafermionic CFTs

– In a QFT, a continuous symmetry G ⟿ 1-FORM conserved current J by NOETHER procedure.

⇒ the charge $Q(\Sigma) = \int_\Sigma {}^*J$ depends on A SURFACE Σ (codimension 1)

But dependence on Σ is TOPOLOGICAL:



$$Q(\Sigma+\delta\Sigma) = Q(\Sigma)$$

⇓

Σ is A TOPLOGICAL SURFACE

— A modern viewpoint on Global Symmetry [Gaiotto, Kapustin, Seiberg, Willet, '14]

"Global Symmetries" $\Longleftrightarrow$ "Codimension 1 Topological Surfaces"

$Q(\Sigma) = \int_{\Sigma} {*J}$ $\qquad$ Σ : $(d-1)$-dim surface.

— A modern viewpoint on Global symmetry [Gaiotto, Kapustin, Seiberg, Willet, '14]

"Global Symmetries" $\leftrightsquigarrow$ "Codimension 1 Topological Surfaces"

$Q(\Sigma) = \int_{\Sigma} {*J}$ $\quad\quad \Sigma$: $(d-1)$-dim surface.

— Advantages:

* higher form sym. $\leftrightsquigarrow$ Codim n Topological surfaces

* high group structure : 't Hooft Anomaly of high-form sym.

* Categorical Sym. $\leftrightsquigarrow$ Non-invertible topological surfaces.

[MANY, e.g. Chang, Lin, Shao, Wang, Yin, 18]

— In 2D CFTs, Codim-1 topological surfaces are

* line objects: commute with stress tensor. $T(z)$ $\bar{T}(\bar{z})$

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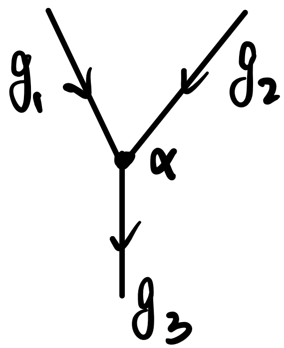

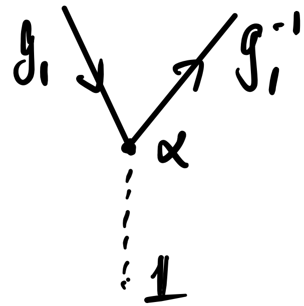

Group multiplication $\leftrightsquigarrow$ Fusion

$$g_1 \cdot g_2 \leftrightsquigarrow \mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}$$

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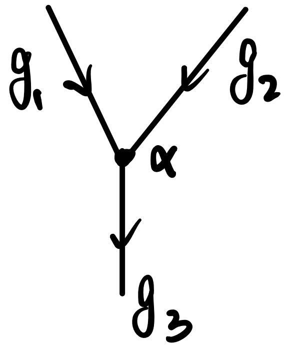
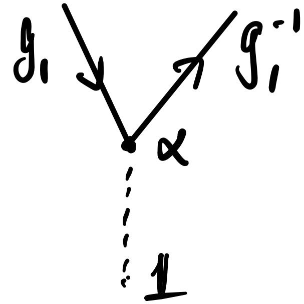
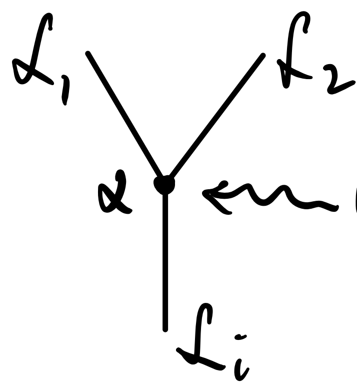




Group multiplication ⇝ Fusion

$g_1 \cdot g_2 \leftrightsquigarrow \mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}$

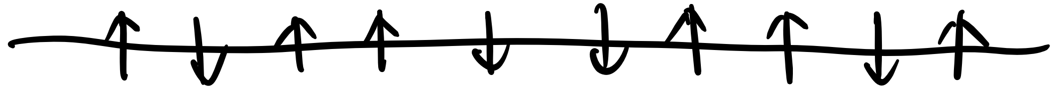
* Categorical sym.



⇜ defect Operator

$$\mathcal{L}_1 \otimes \mathcal{L}_2 = \bigoplus_i n_i \mathcal{L}_i$$

$$\alpha \in \mathrm{Hom}(\mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{L}_i) \simeq \mathbb{C}^{n_i}$$

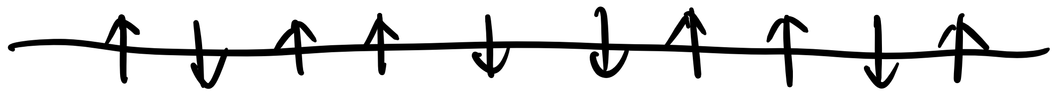
Eg H1d Ising model. 

$$H = -J \sum_i \sigma^z_i \sigma^z_{i+1} - f \sum_i \sigma^x_i \qquad \sigma^{z,x} \text{ – Pauli Matrices.}$$

*. Ground States is two-fold degenracy. $\mathbb{Z}_2$-sym.: $\sigma^z_i \mapsto -\sigma^z_i$

* Two phases parameterized by the order parameter $\langle \sigma \rangle$

Eg 1+1d Ising model.



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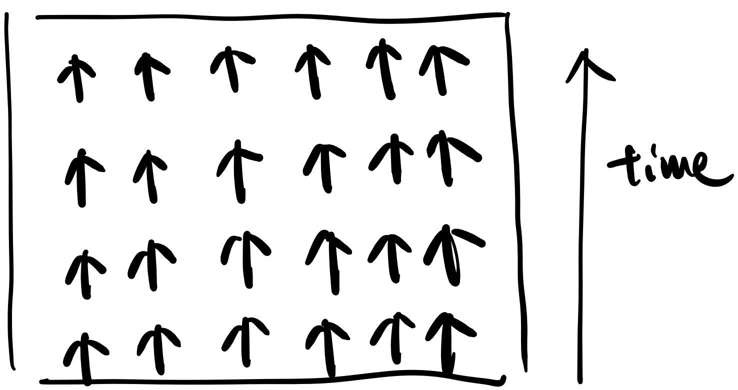
I. $\mathbb{Z}_2$ broken phase (low T)

$\langle\sigma\rangle \neq 0$ $\qquad \lambda > 1$



$\langle \sigma_0 \sigma_n \rangle \sim n^{-\Delta_\sigma}$

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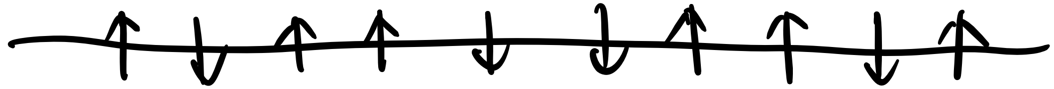

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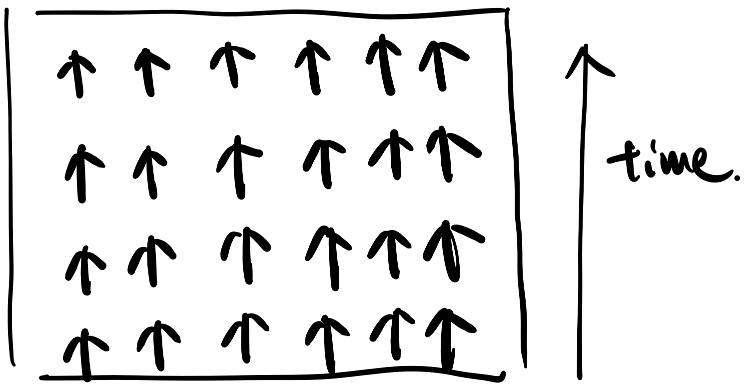

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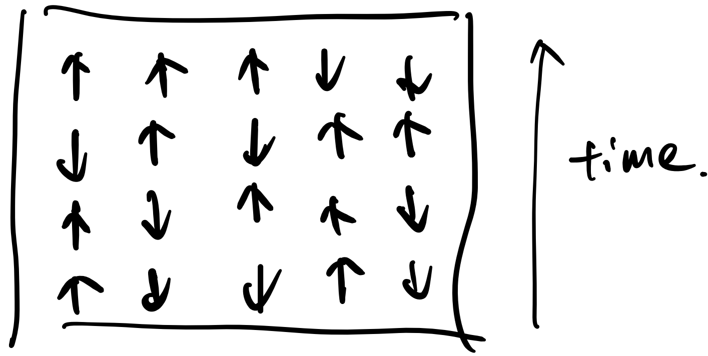

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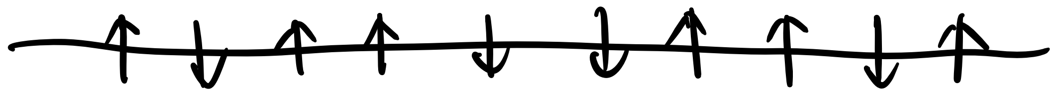

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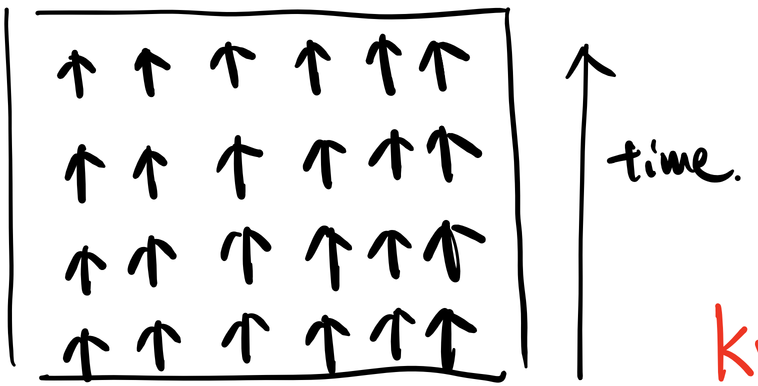

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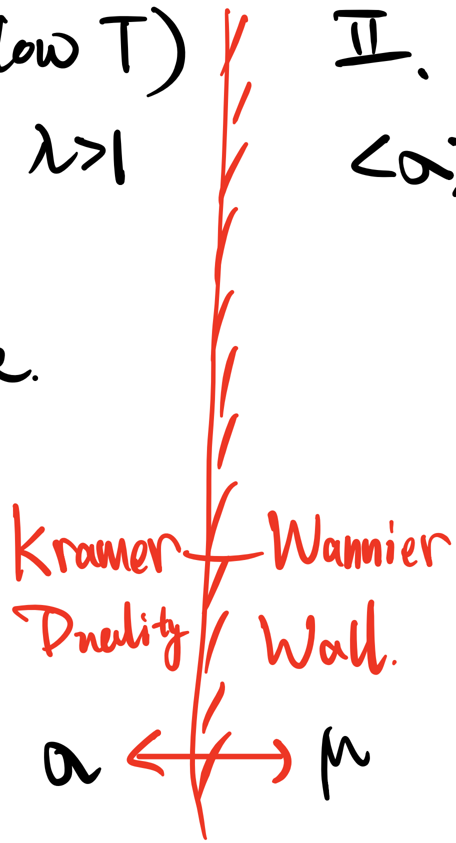

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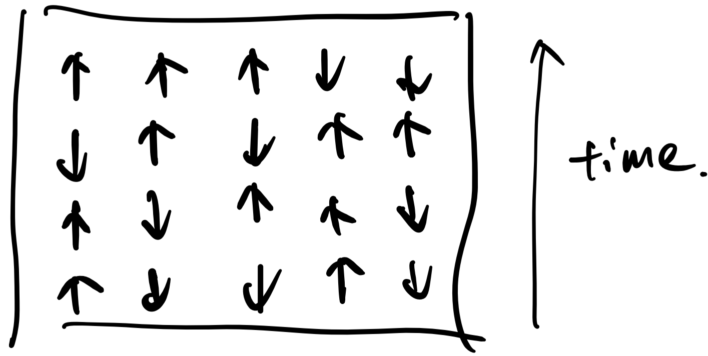

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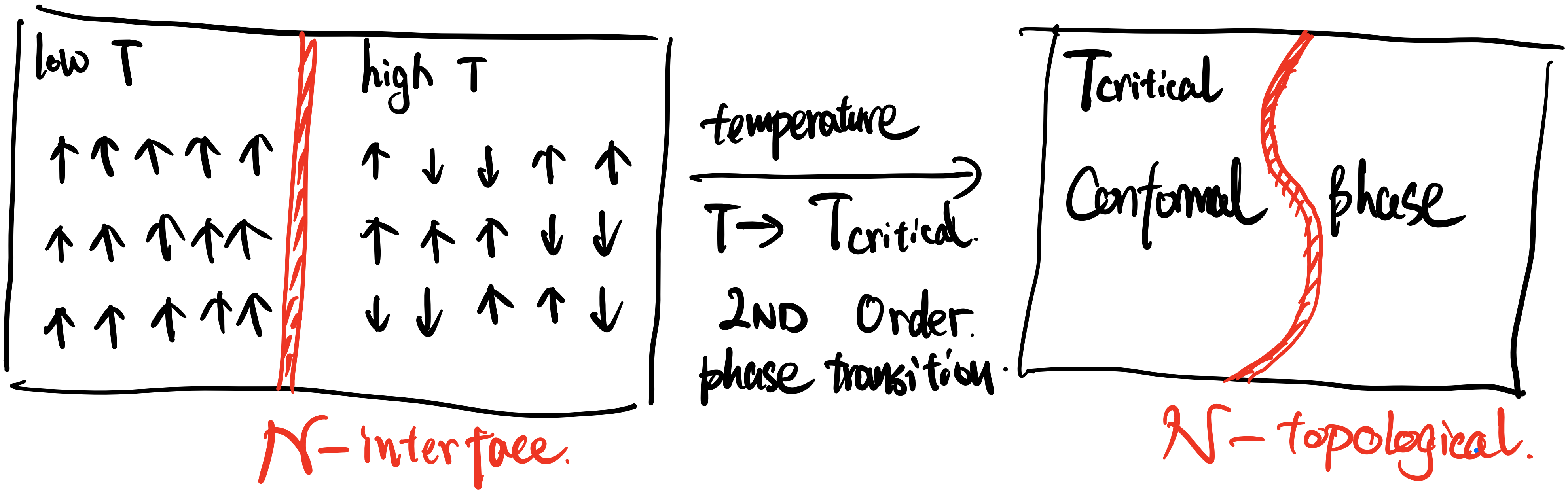
– CRITICAL ISING.



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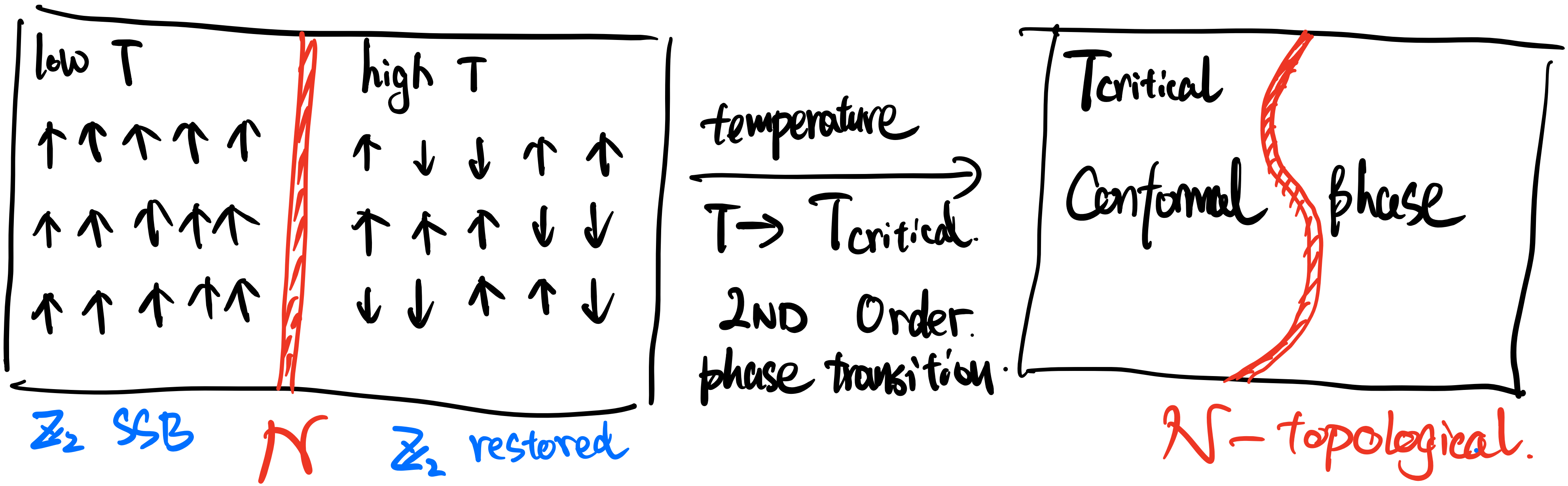
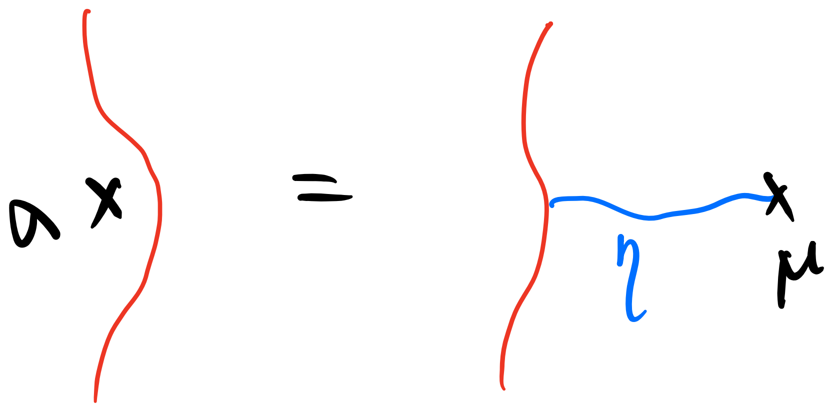


N – Non-invertible duality line (Duality Domain wall)

swapping ordered(σ)/disordered(μ) Operators.

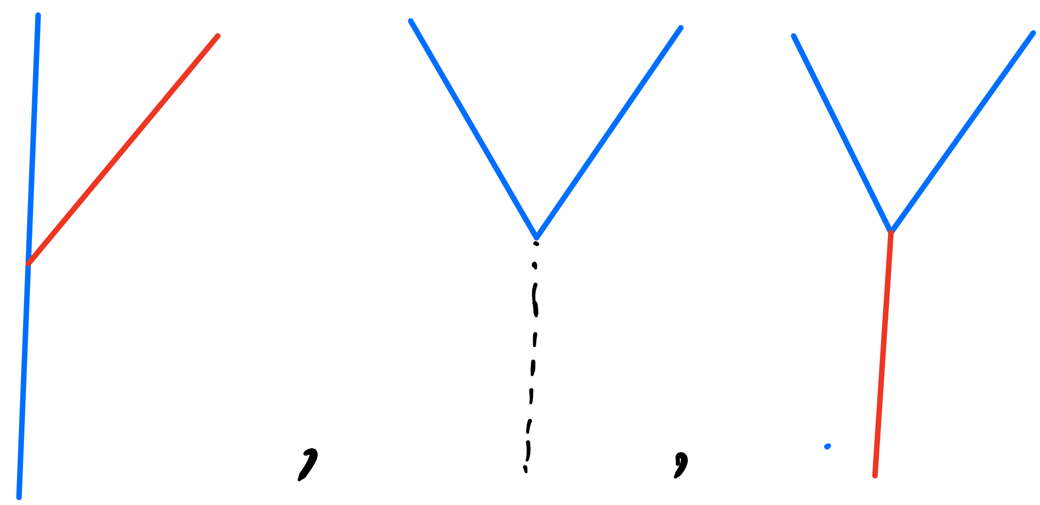


σ – local operator

μ – quasi-local vortex operater.

* $\{\mathbb{1}, \eta, N\}$ FURNISH the Ising Category

Fusion	$\mathbb{1}$	η	N
$\mathbb{1}$	$\mathbb{1}$	η	N
η		$\mathbb{1}$	N
N			$\mathbb{1}+\eta$



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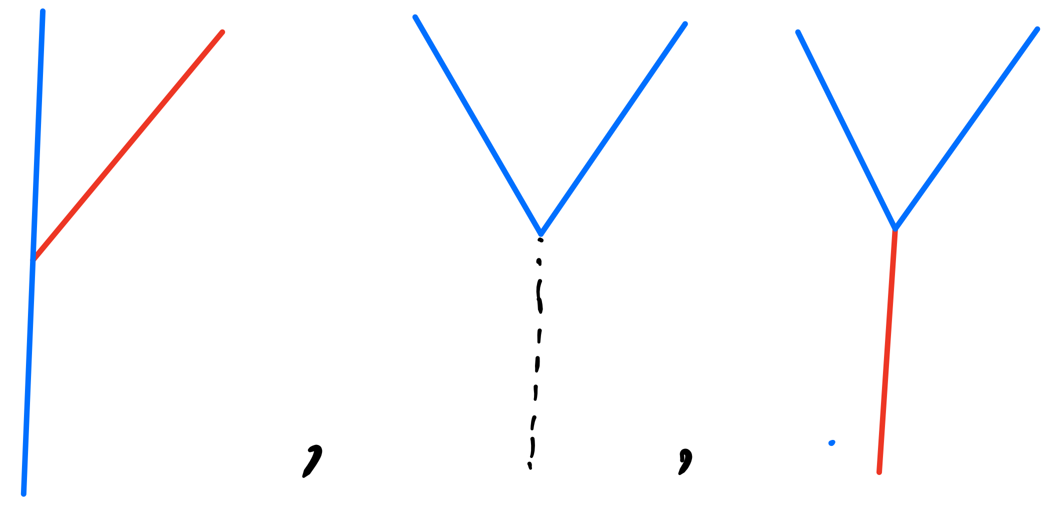

* IN 2D RATIONAL CFTs, every primary $\phi_i \longleftrightarrow$ TDL $\mathcal{L}_i$

$\mathcal{L}_i$ – VERLINDE LINE

$$\mathcal{L}_i \otimes \mathcal{L}_j = \sum_k \mathcal{N}_{ij}^{k} \mathcal{L}_k$$

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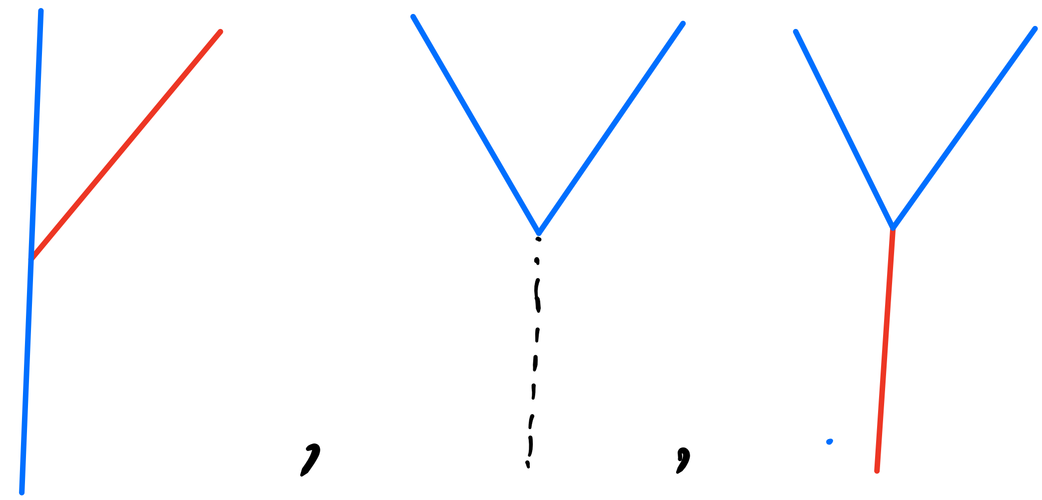

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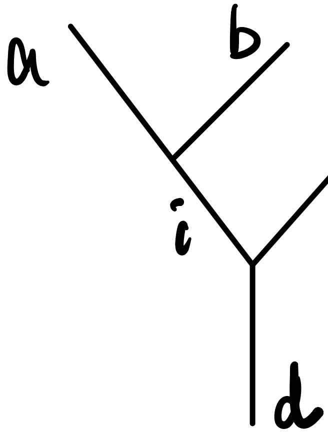
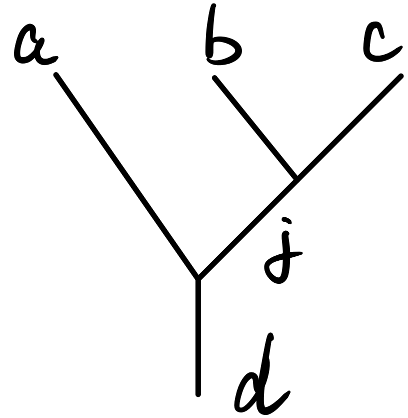
* FOR A closed set $\mathcal{C}$ with FINITE # of TDLs,

$\mathcal{C}$ furnishes A **FUSION CATEGORY**

– F-SYMBOLS. There are phase ambiguities for 3 TDLs' Fusion.



$$= \sum_j \tilde{F}^{abc}_d(i,j)$$



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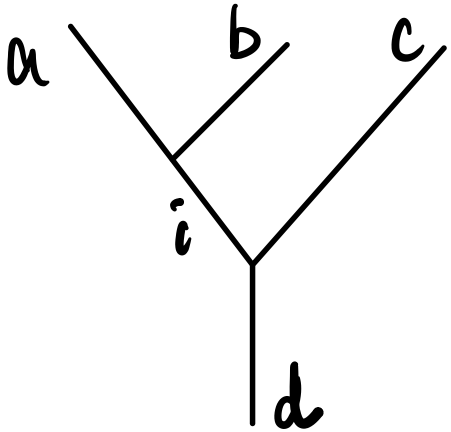

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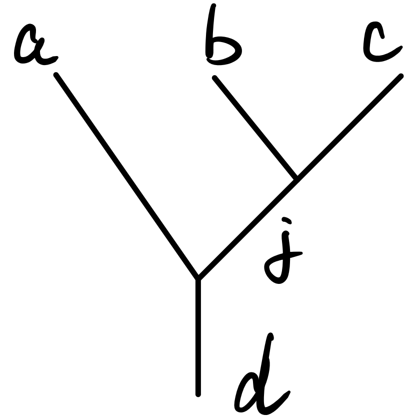

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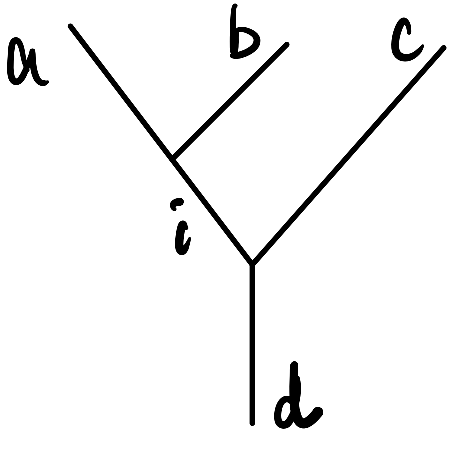

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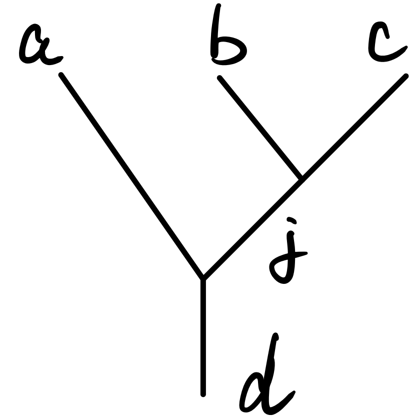

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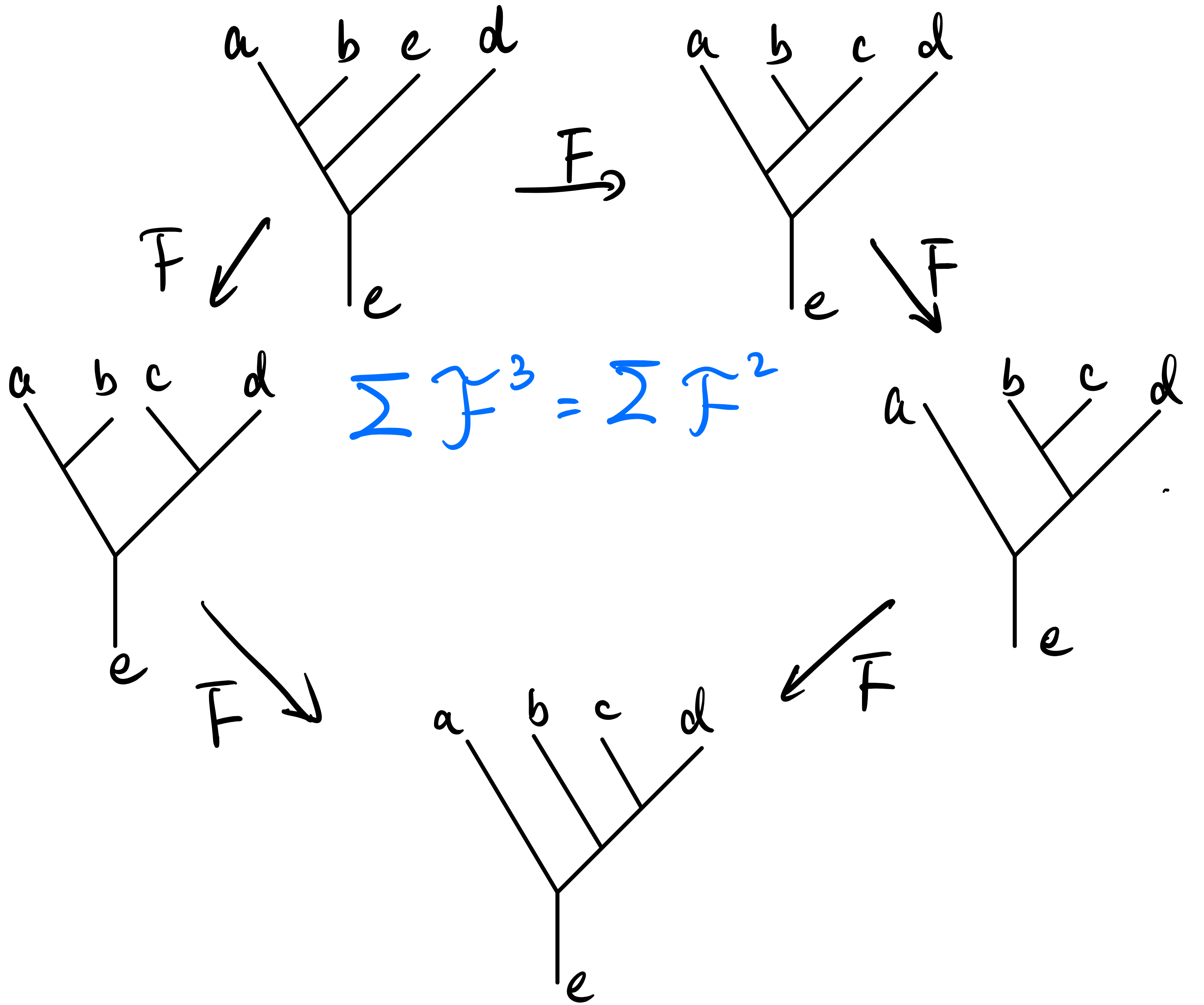
* A Fusion CATEGORY $\mathcal{C}$ is uniquely specified by A set of F-symbols.

— Pentagon Identity



$$\sum F^3 = \sum F^2$$

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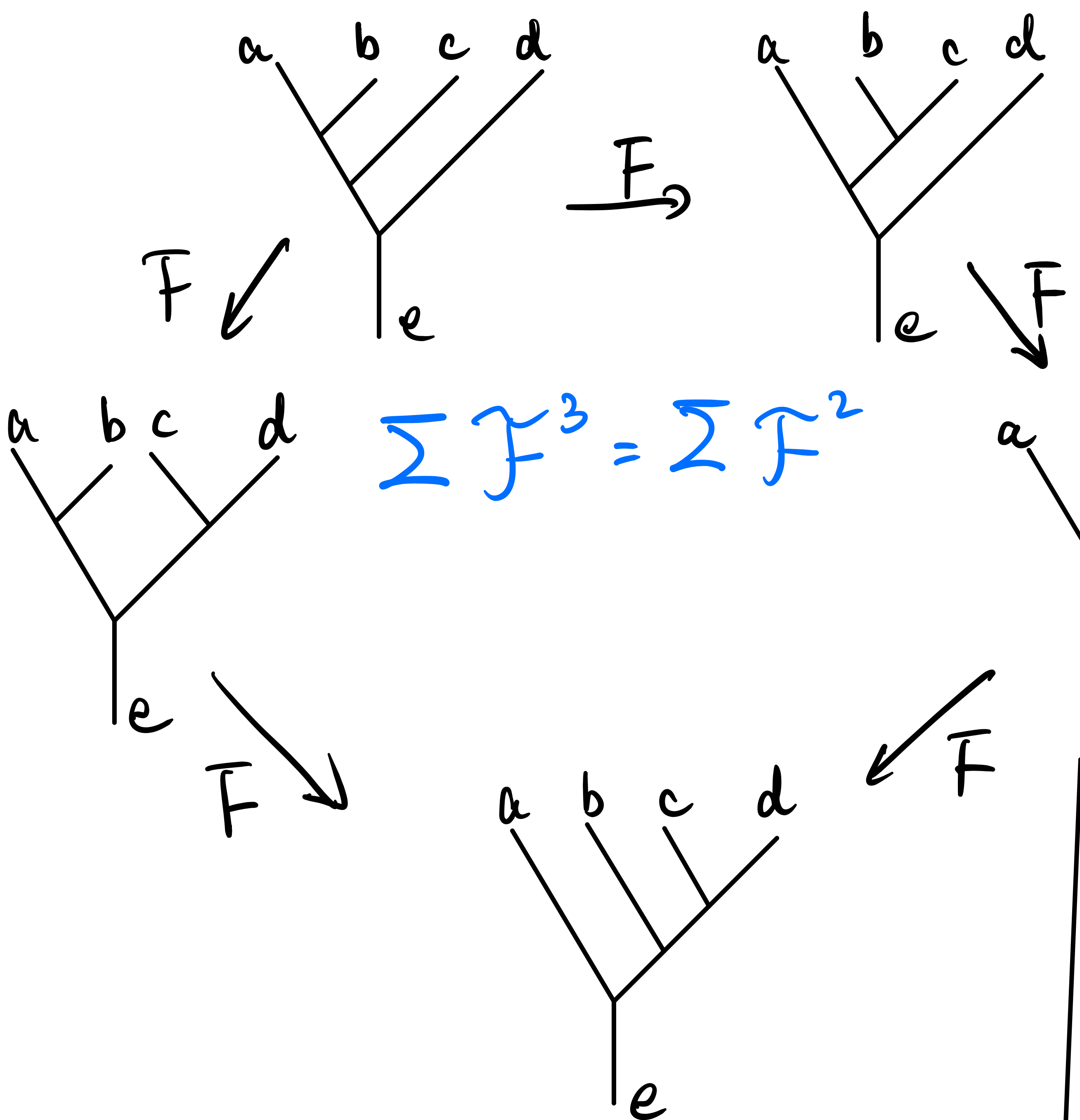


$$\sum \mathcal{F}^3 = \sum \mathcal{F}^2$$

Solving all $\{\mathcal{F}\}$

⇝ classify $\mathcal{C}$.

sol. is Finite.

(Ocneanu Rigidity)

⇝ Invariant under RG-Flow

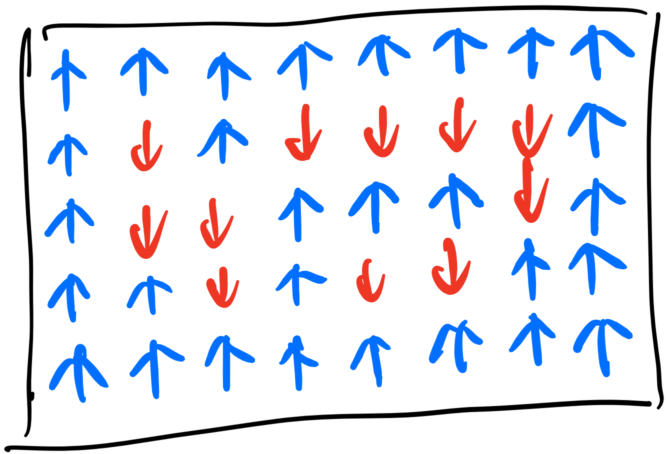
For $\mathcal{C} = \mathrm{Vec}_G^{\alpha}$

sol. of Pentagons

⟺ $\alpha \in H^3(G, U(1))$

– Parafermionization / Orbifold resp. para-spin structure ρ.

e.g. Ising:



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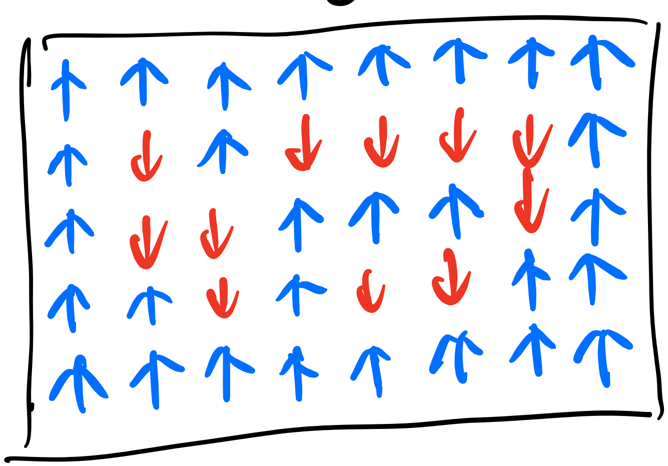

local D.O.F. : $\uparrow$ or $\downarrow$

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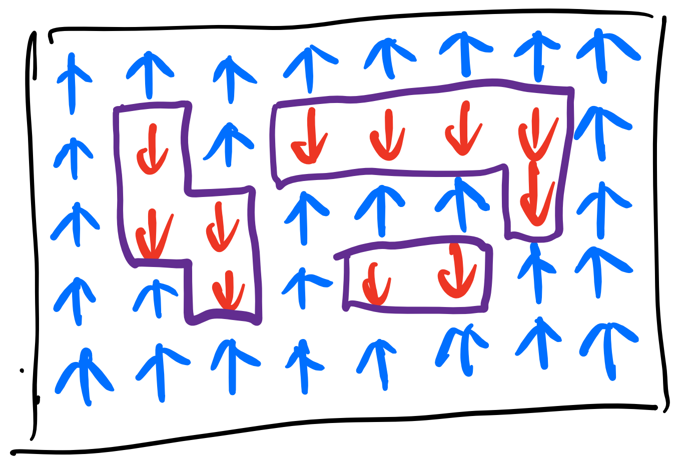
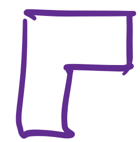


Non-local Domain Wall: 

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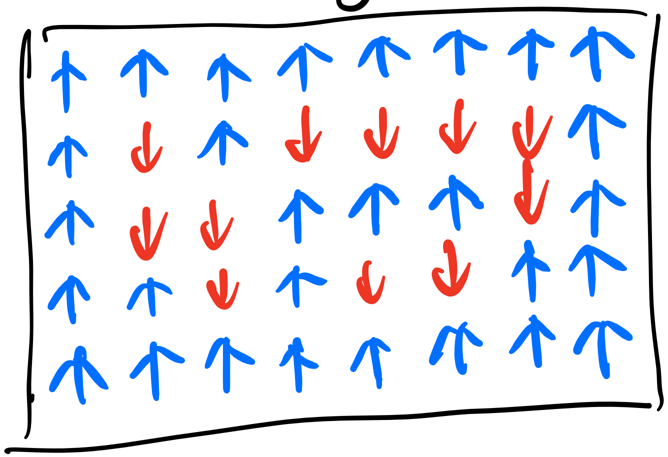
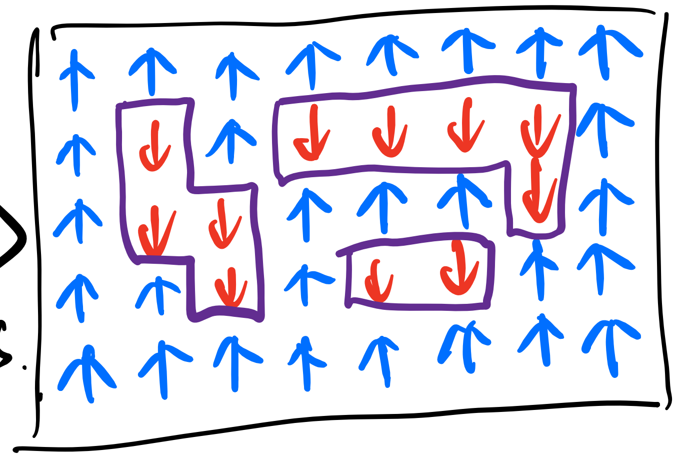


JORDAN — WIGNER Trans.



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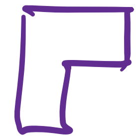
Non-local Domain Wall: 

Ising

Fermionization

Bosonization.

Maj. Fermion 

In the Ising

σ: •

μ: ×———

In the Majorana.

ψ: •〰〰 ≡ ×———

	PHASE A	PHASE B.
	$\lambda > 1$	$\lambda < 1$
	$\mathbb{Z}_2$-SSB	$\mathbb{Z}_2$ restore.
	$m = \frac{1-\lambda}{a\lambda} < 0$	$m > 0$
	∃ Maj. ZERO MODES.	No Maj. ZERO MODES.

[KITAEV, 2001]

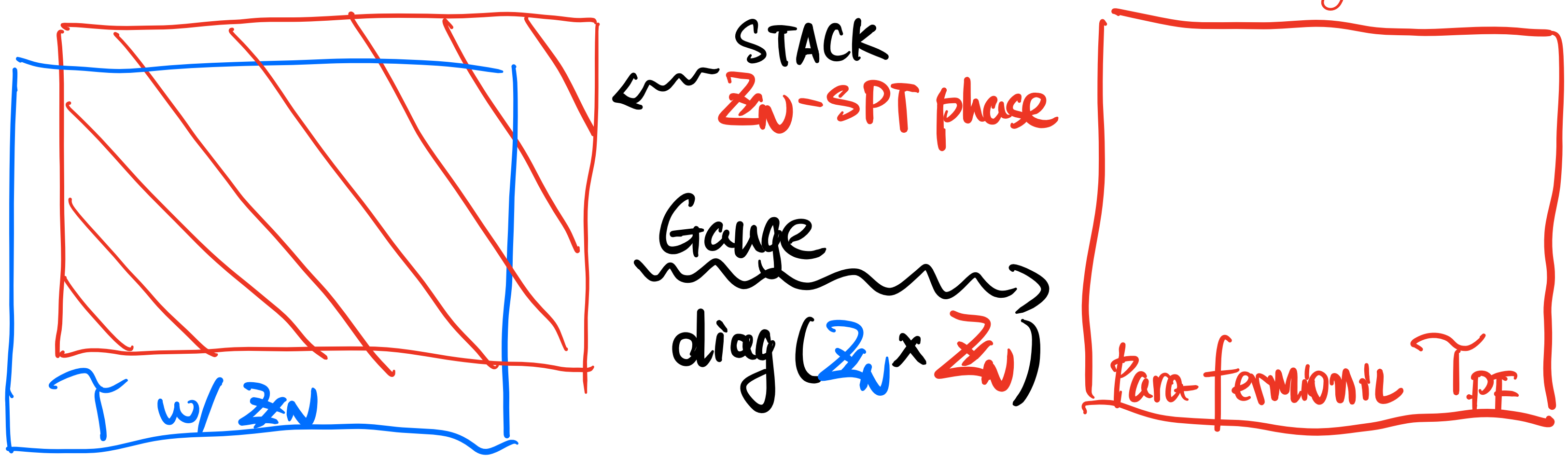
	PHASE A $\lambda > 1$	PHASE B $\lambda < 1$
In the Ising σ: • μ: 	$\mathbb{Z}_2$-SSB	$\mathbb{Z}_2$ restore.
In the Majorana. ψ: 	$m = \frac{1-\lambda}{a\lambda} < 0$ ∃ Maj. ZERO MODES. [KITAEV, 2001]	$m > 0$ No Maj. ZERO MODES.

Some Comments:

1. Maj. ZERO MODES define Invertible fermionic topological order (IFTO, $\mathbb{Z}_2$-SPT)
2. In QFT, $Z_{IFTO} = (-1)^{\mathrm{Arf}_2(P)} = (-1)^{\mathrm{Ind}(\not{D}_P)}$ [KARCH, TONG, TURNER, 2019]
   Measuring the parity of the # of Maj. ZERO MODES
3. IFTO can be used to fermionize Boson system.

* Generically, one prepares a $\mathbb{Z}_N$-SPT phase [Fendley, 2012]

* Stack onto a Bosonic CFT $\mathcal{T}$ with $\mathbb{Z}_N$-sym. [Hsieh, Nakayama, Tachikawa, 2020]



* Gauge diag$(\mathbb{Z}_N \times \mathbb{Z}_N)$ ⇝ $\mathcal{T}_{PF} = \dfrac{\mathcal{T} \otimes \mathbb{Z}_N\text{-SPT}}{\text{diag}(\mathbb{Z}_N \times \mathbb{Z}_N)}$

$\mathcal{T}_{PF}$ is sensitive to para-spin structure [Thorngren, Wang, 2021] [Yao, Furusaki, 2021]

– TDLs in Para-Fermionic CFTs (pf-CFTs)

* A distinguish feature for TDL in pf-CFTs from bosonic CFT is [Gu, Wang, Wen, 15] [JC, Haghighat, Wang, 23] [Zhou, Wang, Gu, 21] [Duan, Jia, Lee, 23]

There are defect operator of fractional spin that can "live" on TDLs.

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> There are **defect operator of fractional spin** that can "live" on TDLs.

e.g. Ising Model $\rightsquigarrow$ Majorana fermion. $(N=2)$

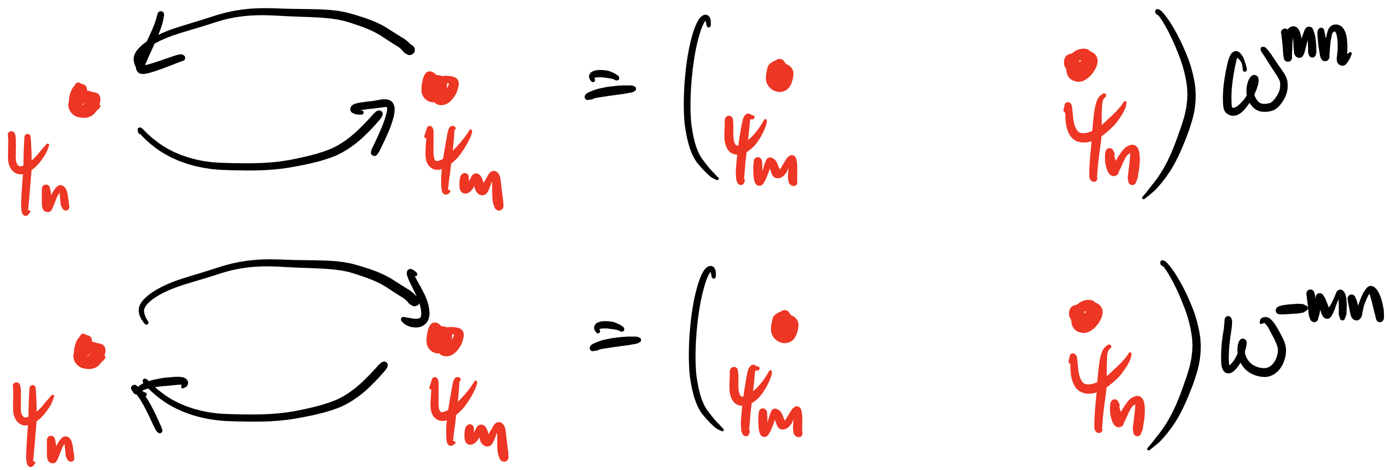
$\exists\ \psi_{0,1}$ $\frac{1}{2}$-spin

3-Pott Model $\rightsquigarrow$ $\mathbb{Z}_3$ parafermion $(N=3)$

$\exists\ \psi_{0,1,2}$ $\frac{2}{3}$-spin.

* Because of FRACTIONAL SPINS

These ψ_n satisfy NOVEL statistics.



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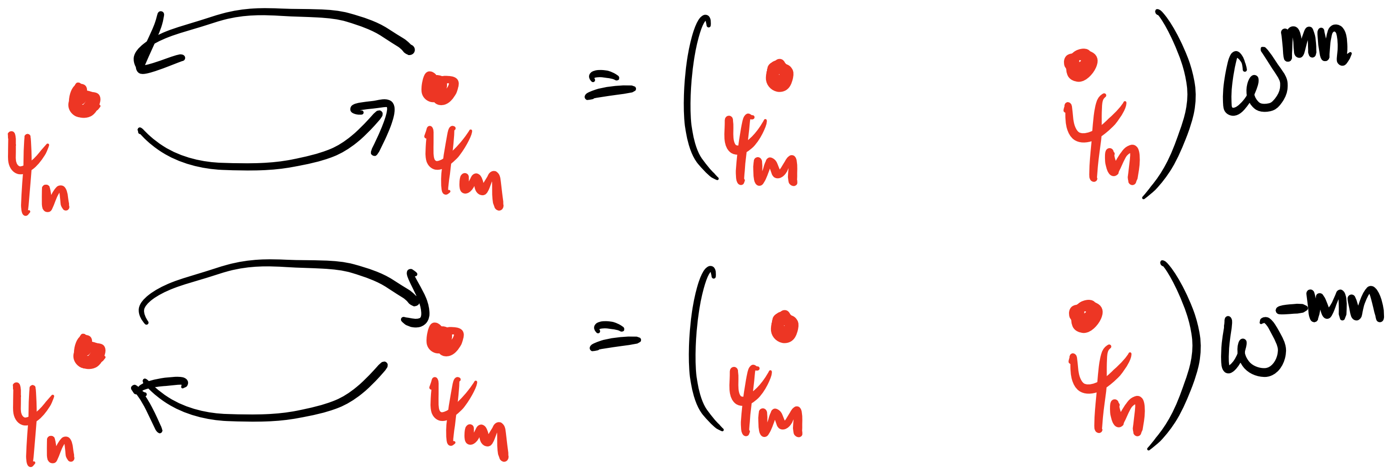
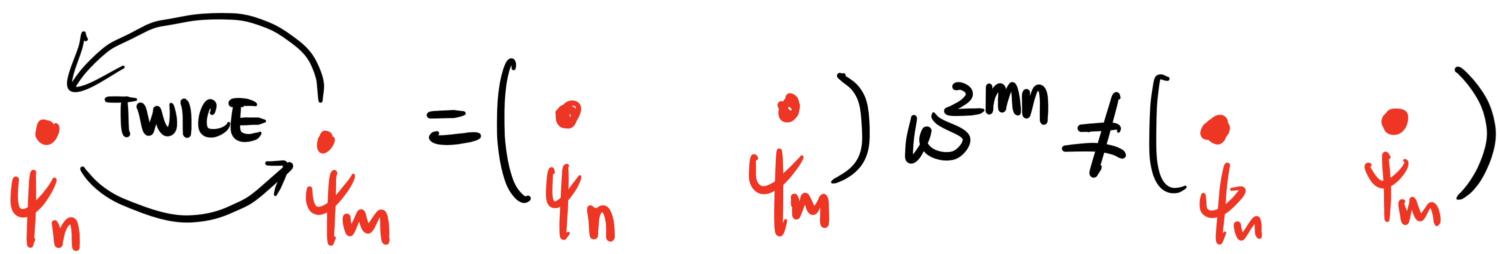


For $m, n \in \mathbb{Z}_N$.

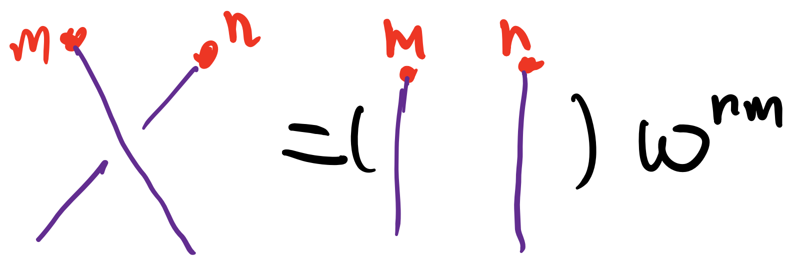
* For $N=2$, $\omega = e^{\pi i} = -1$ ⇝ ψ_1 is LOCAL Operator.

For $N \geq 3$, $\omega = e^{\frac{2\pi i}{N}}$ ⇝ ψ_n is Quasi-LOCAL



* It's LIKE ψ_n's are ENDING on Strings.

String admits a BRAIDING



For N=2 2-Braiding is trivial. ⇝ ψ_1 LOCAL.

II. An 3D Anyon Approach

- Start from a 2D CFT with .
  Fusion $\mathcal{C} \rightsquigarrow Z(\mathcal{C})$, Drinfeld Center

Fusion Cat on Boundary $\xrightarrow{\text{Turaev-Viro/Levin-Wen}}$ Modular Tensor Cat on Bulk

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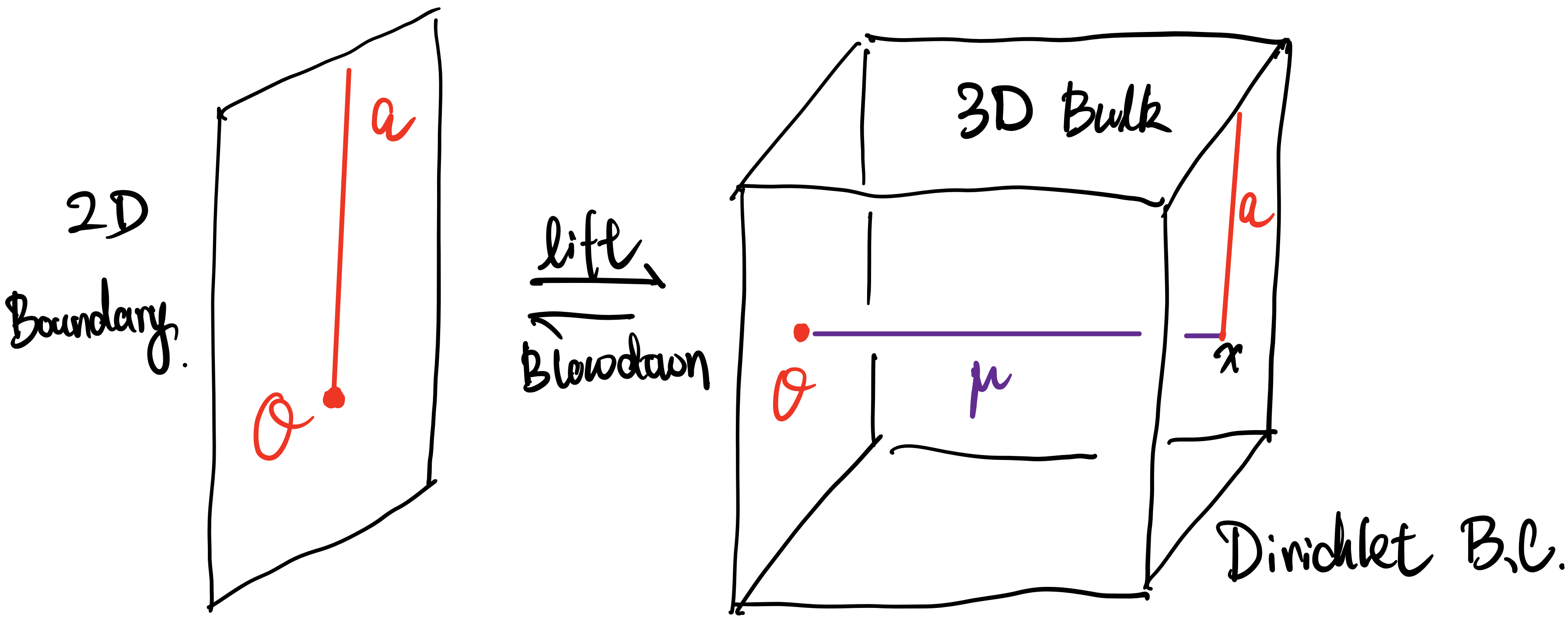
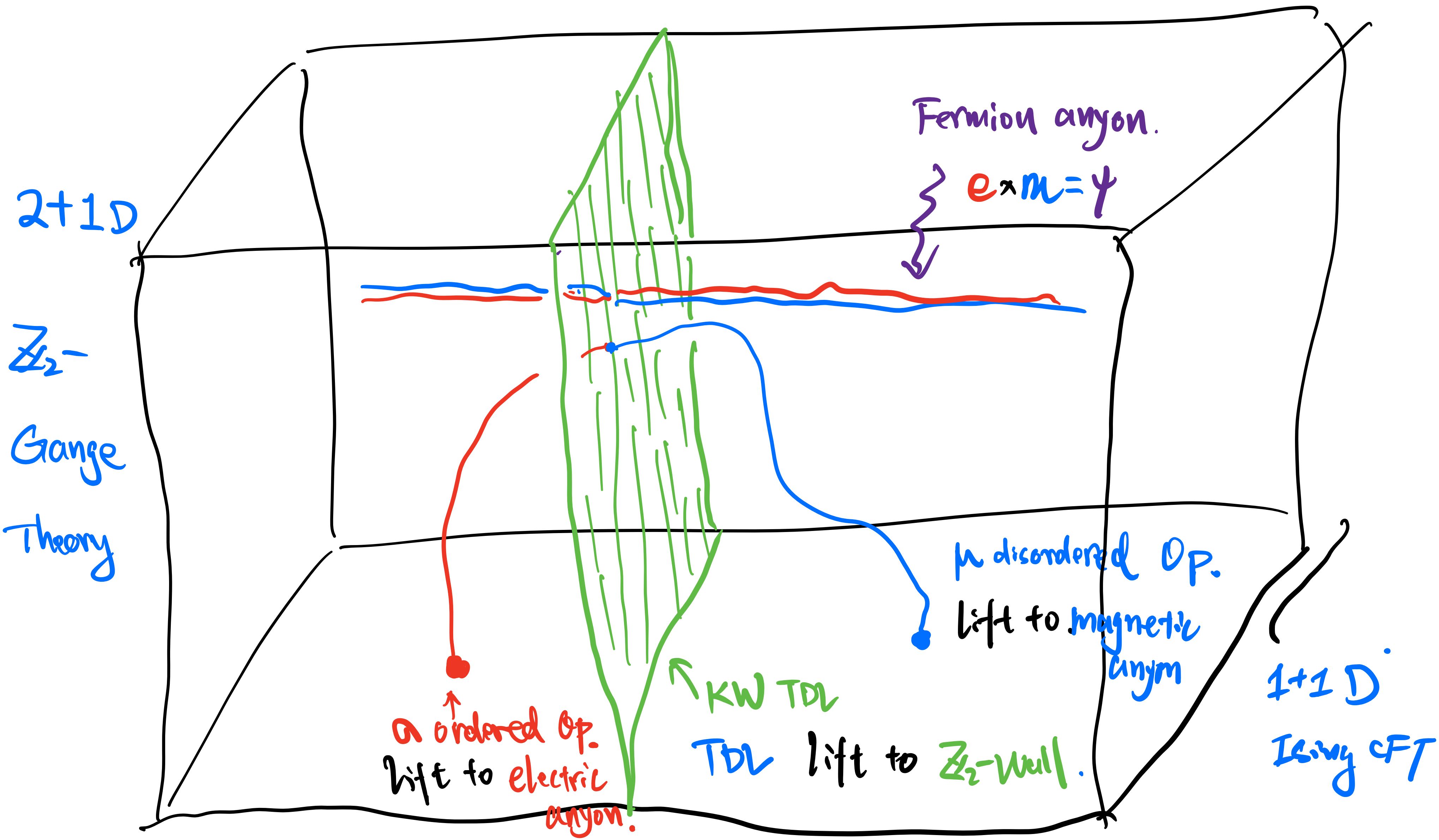
Fusion CAT on Boundary	TURAEV-VIRO/ LEVIN-WEN ⇝	MODULAR TENSOR CAT on Bulk



Eg. Ising model: using $\mathcal{C} = Vec_{\mathbb{Z}_2}$

$Z(\mathcal{C})$ is 2+1 D $\mathbb{Z}_2$ - Dijkgraaf Witten theory.



* Boundary Data can be mapped by Forgetful Functor.

$$F: Z(\mathcal{C}) \longrightarrow \mathcal{C}.$$

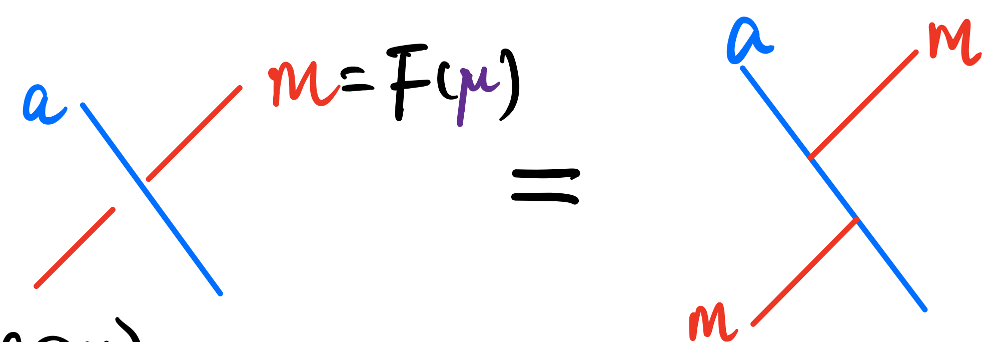
* Bulk Data can be intrinsically described by.

half-braidings:

$R_{a,m}$:

$\Uparrow$

$\mathrm{Hom}(m \otimes a, a \otimes m)$



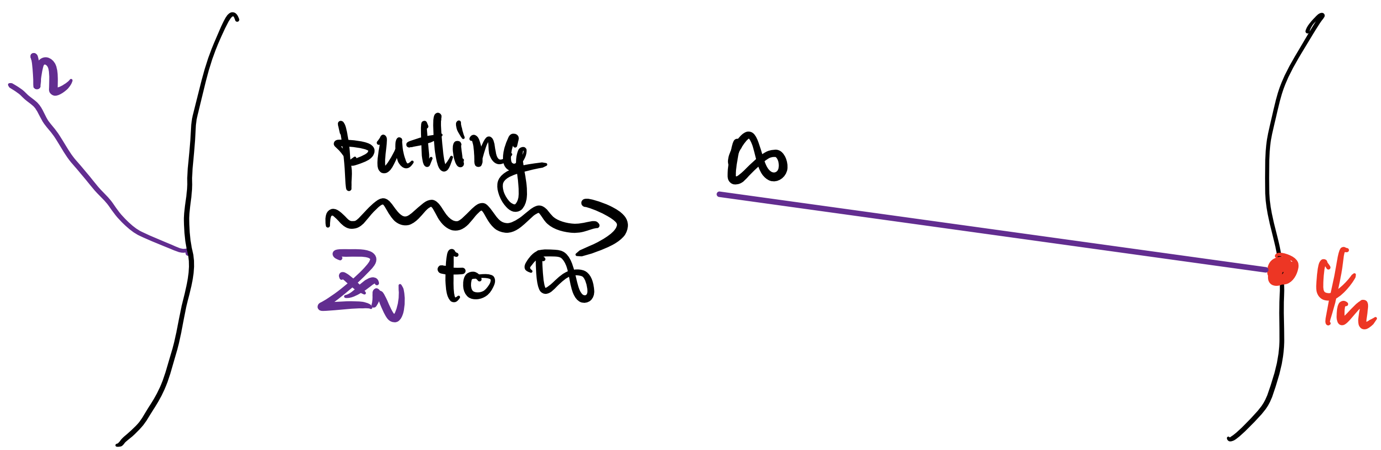
— In this set-up:

Gauge $\mathbb{Z}_N$ TDLs on Boundary. ⇝ ① lift to Bulk.

② (para-fermionic) condense $\mathbb{Z}_N$-anyon.

③ map back to Boundary.

Condensing:



Consider the following configuration:

Consider the following configuration:

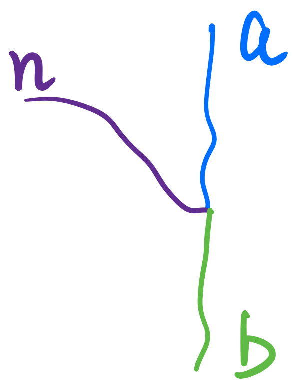
The fractional defect Op. $\{\Psi_n\}$ is the remeniscent of $\mathbb{Z}_N$-anyons: Nontrivial statistics ⟿ BRAIDING of $\mathbb{Z}_N$-anyons

— q-TYPE & m-TYPE TDLs in pf-CFTs.

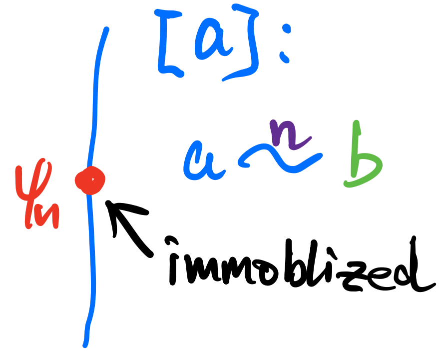
* m-type:

$a \cdot n = b$

Before



After.

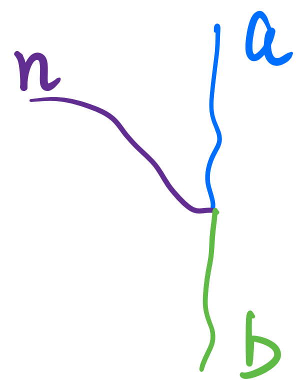


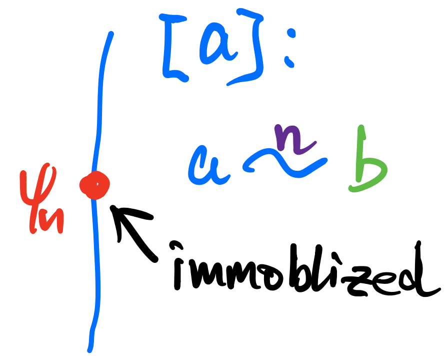
— q-TYPE & m-TYPE TDLs in pf-CFTs.

Before | After

* m-type:

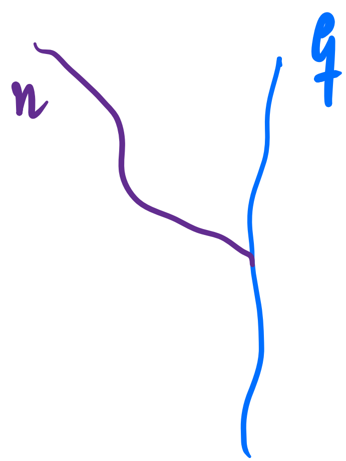
$a \cdot n = b$

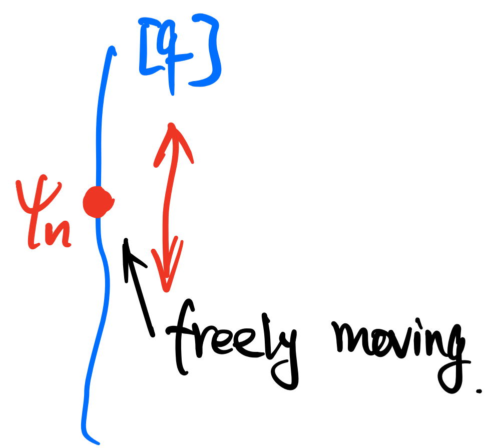




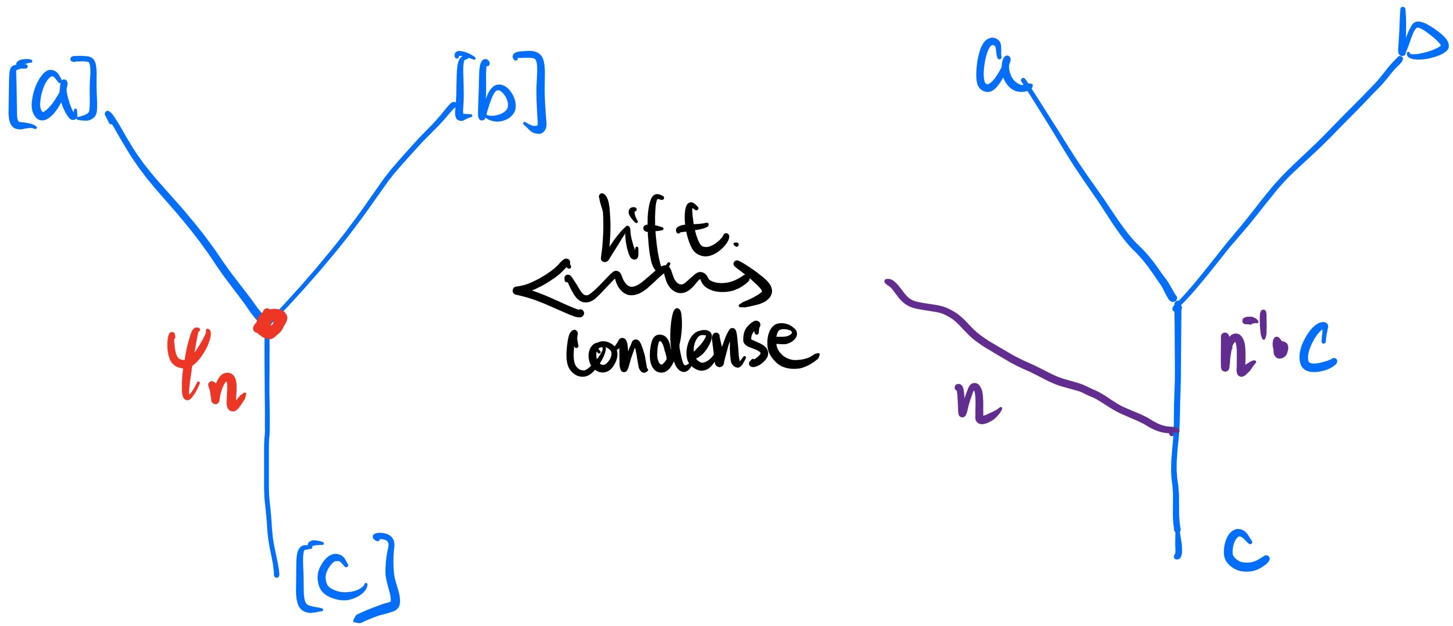
* q-type:

$q \cdot n = q$





— Fusion of TDLs in bf– CFTs.



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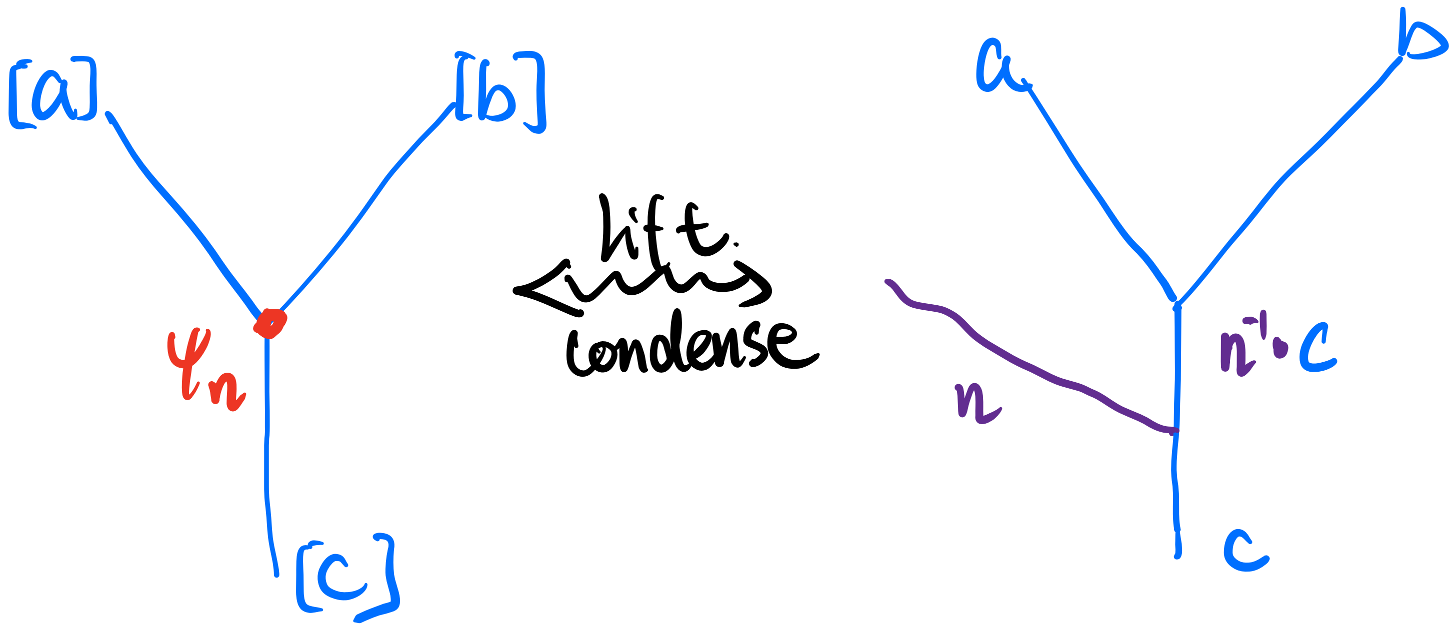
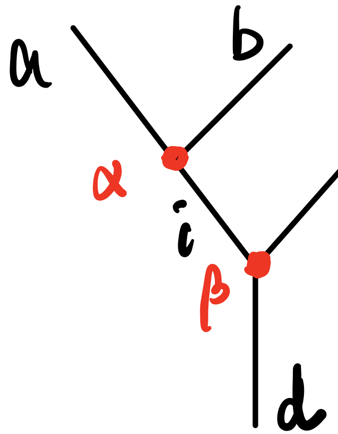


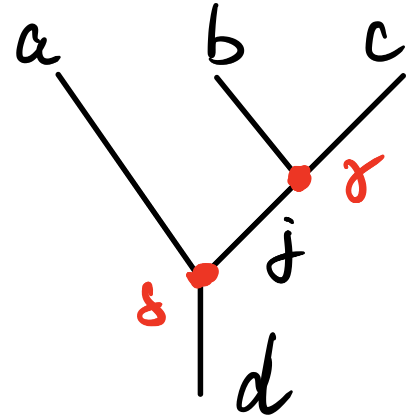
$$[a]\cdot[b] = \sum_{c} \underbrace{\mathbb{C}^{i_1|i_2|\cdots i_d}}\,[c]$$

$i_k \in \mathbb{N}$ label the type of fractional defect that can live on the junction dubbed "colors"

— F-SYMBOLS IN pf-CFTs.



$$= \sum_{\delta,\gamma} \sum_{j} \left(\tilde{F}^{abc}_{d}(i,j)\right)^{\alpha\beta}_{\gamma\delta}$$



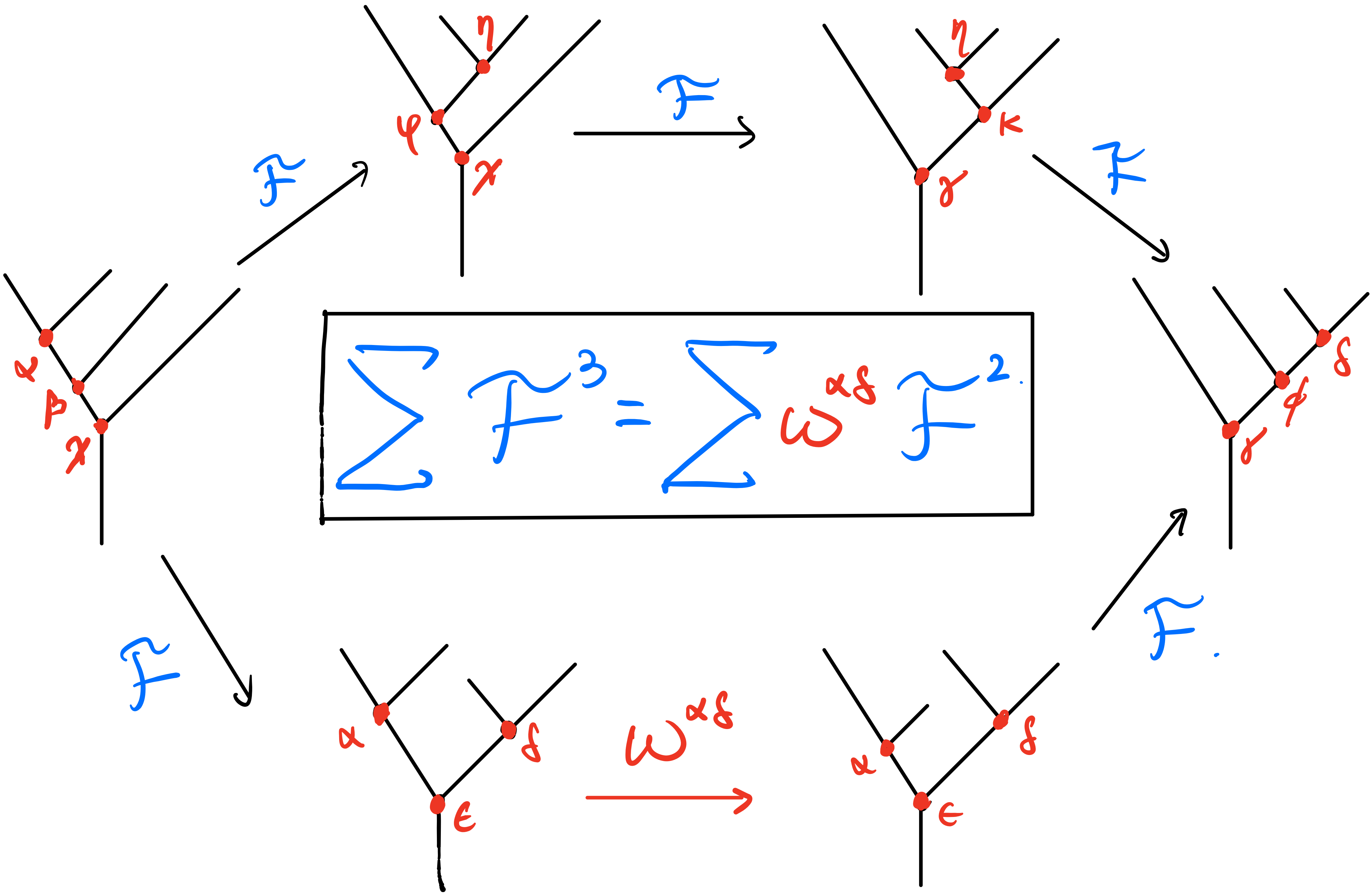
$\alpha, \beta, \gamma, \delta$ are $\mathbb{Z}_N$-VALUED, labeling fractional defect.

Selection rule: $\tilde{F} = 0$

(Due to pf-parity)

if $\alpha + \beta \neq \gamma + \delta \pmod{N}$

— PARA - PENTAGONS.



$$\sum F^3 = \sum \omega^{\alpha\delta} F^2.$$

— $\mathbb{Z}_N$ PARA-FUSION CATEGORY. [JC, Haghighat, Wang, 23]

* Objects: 1d TDLs

- m-type: $\mathbb{Z}_N$-transitive
- q-type: $\mathbb{Z}_\ell$-fix pts. $\mathbb{Z}_\ell \subset \mathbb{Z}_N$ ($\ell | N$, $\ell \neq 1$)

— $\mathbb{Z}_N$ Para-Fusion Category. [JC, Haghighat, Wang, 23]

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* Morphism: 0d fractional Defect Operators.

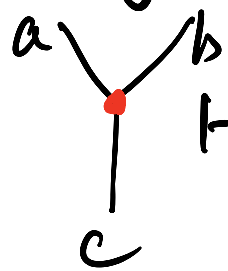
m-type

$\mathrm{End}(m) = \phi$

q_ℓ-type

$\mathrm{End}(q_\ell) = \mathbb{C}^{i_1 \cdots i_\ell}$

3-way Junction



$\mathrm{Hom}(a \cdot b, c) = \mathbb{C}^{i_1 \cdot i_m}$

— $\mathbb{Z}_N$ PARA-FUSION CATEGORY. [JC, Haghighat, Wang, 23]

* Objects: id TDLs $\begin{cases} m\text{-type}: \mathbb{Z}_N\text{-transitive} \\ q\text{-type}: \mathbb{Z}_\ell\text{-fix pts.} \\ \qquad \quad \cap \\ \qquad \ \ \mathbb{Z}_N \quad (\ell | N, \ \ell \neq 1) \end{cases}$

* Morphism: od fractional Defect Operators.

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$\mathrm{End}(q_\ell) = \mathbb{C}^{i=1\cdots l}$

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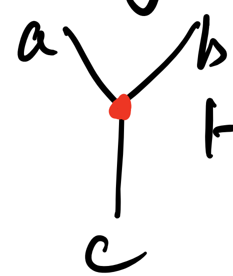


$\mathrm{Hom}(a \cdot b, c) = \mathbb{C}^{i=1\cdots N}$

* F-SYMBOL & PARA-PENTAGON.

III. Examples:

— Warm-up (Bosonic condensation)

$m=5$ A-type minimal CFT: 10 primary ⇝ 10 TDLs.

$\mathbb{Z}_2$-sym. (on x)

$$\mathcal{C} = \mathrm{Fib} \boxtimes \{\underbrace{\mathbb{1}, x, y}_{\mathrm{Rep}(S_3)}, n_1, n_2\}$$

III. Examples:

— Warm-up (Bosonic condensation)

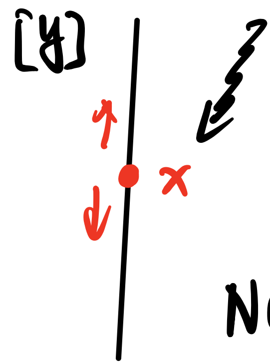
$m=5$ A-type minimal CFT: 10 primary ⇝ 10 TDLs.

$$\mathcal{C} = \mathrm{Fib} \boxtimes \{\underbrace{\mathbb{1}, \color{red}{x}, y}_{\mathrm{Rep}(S_3)}, n_1, n_2\}$$

(x ⇝ $\mathbb{Z}_2$-sym.)

Condense μ_x-anyon, $\mu_{\mathbb{1}} \sim \mu_x$, $\mu_{n_1} \sim \mu_{n_2}$, μ_y - fixed pt.

map back to CFT,



bosonic defect. $h_x = 3$.

$[\mathbb{1}]$, $[y]$, $[n]$

$\Rightarrow \mathrm{End}([y]) = \mathbb{C}^2$

NOT SIMPLE ⇝ $[y] \to [y_1]$, $[y] \to [y_2]$

* $\{[1\!\!1], [y_1], [y_2]\}$ Furnish a $\mathbb{Z}_3$-sym., $(\mathrm{Rep}(S_3)/\mathbb{Z}_2 = \mathbb{Z}_3)$

* $\mathrm{Fib} \boxtimes \mathbb{Z}_3 \rightsquigarrow$ {Verlinde Lines} Resp. W_3-alg.

* $[n]\cdot[n] = [1\!\!1] + [y_1] + [y_2]$

$$\rightsquigarrow \{[1\!\!1], [y_1], [y_2], [n]\} \simeq TY(\mathbb{Z}_3)$$

* Counting the emergent $\hat{\mathbb{Z}}_2$-sym., $(\mathbb{Z}_3 \rtimes \hat{\mathbb{Z}}_2 \simeq S_3)$

* $m=5$ D-type CFT after $\mathbb{Z}_2$-gauge of A-type.

$${}_A\mathcal{C}_A \simeq \underset{2}{\mathrm{Fib}} \boxtimes \underset{\times\ 4}{TY(\mathbb{Z}_3)} \boxtimes \underset{\times\ 2}{\hat{\mathbb{Z}}_2} \rightsquigarrow \text{16 TDLs.}$$

— Para-Tambara-Yamagami $\mathbb{Z}_N$, $pfTY(\mathbb{Z}_N)$ [JC, Haghighat, Wang, 23]

Noninvertible.

* $TY_{\mathbb{Z}_N} = \mathbb{Z}_N \cup \{\mathcal{N}\}$, such that

$$\mathcal{N} = a \otimes \mathcal{N} = \mathcal{N} \otimes a, \qquad \mathcal{N}^2 = \sum a$$

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$*$ It's A <u>UNIVERSAL Fusion CAT</u> for CFT $\begin{cases} \mathbb{Z}_N\text{-sym.} \\ \mathcal{N}\text{-selfdual} \end{cases}$

e.g. Ising $(N=2)$, 3-Potts $(N=3)$, Compact Boson

$SU(2)_N/U(1)$ coset WZW ...

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e.g. Ising ($N=2$), 3-Potts ($N=3$), Compact Boson

$SU(2)_N/U(1)$ coset WZW ....

* $TY^{t,k}_{\mathbb{Z}_N}$ is characterized by TWO parameters t, k

t – Define bi-linear form $\chi_t(a,b) = e^{\frac{2\pi i}{N} tab}$.

k – Frobenius–Shur Indicator $k = \pm 1$.

* para-fermionizations of these $\mathbb{Z}_N$ selfdual CFT

⇓

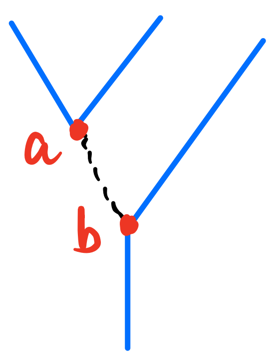
pf-CFTs also ADMIT a universal $\mathbb{Z}_N$-para-fusion CAT

$$\text{pf-}TY_{\mathbb{Z}_N} = \{ [\mathbb{1}], [\mathcal{N}] \}$$

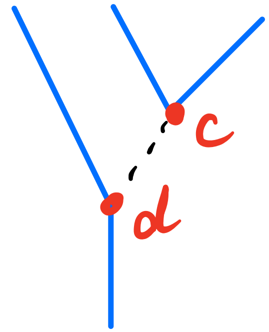
with Fusion rule

$$[\mathcal{N}] \otimes [\mathcal{N}] = \underbrace{\mathbb{C}^{1|1|\cdots|1|1}}_{n\text{-colors.}} [\mathbb{1}]$$

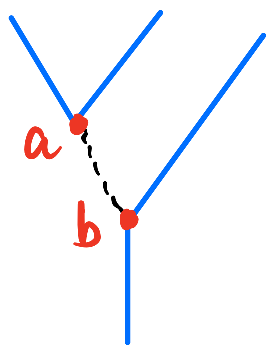
*

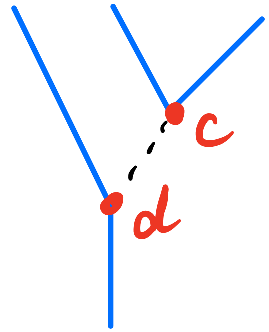


$$= \sum_{cd} \mathcal{F}^{ab}_{cd}$$



$\mathcal{F}$ — $N^2 \times N^2$ matrix satisfy para-pentagons.

* 

$= \sum_{cd} F^{ab}_{cd}$ 

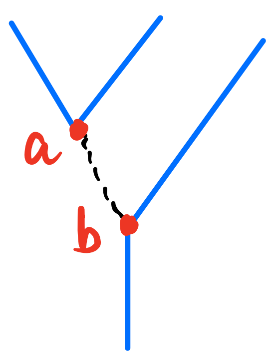
F — $N^2 \times N^2$ matrices

* Half-Braiding assign a map from

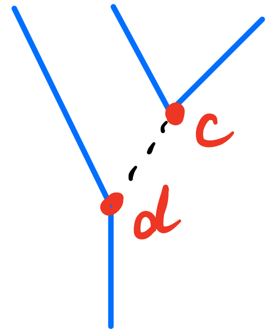
$$\{F\}_{TY^{t,K}_{\mathbb{Z}_N}} \rightsquigarrow \{F\}_{\text{pf-}TY^{t,K,B}_{\mathbb{Z}_N}}$$

It automatically solve all para-pentagons. !

*



$= \sum_{cd} F^{ab}_{cd}$

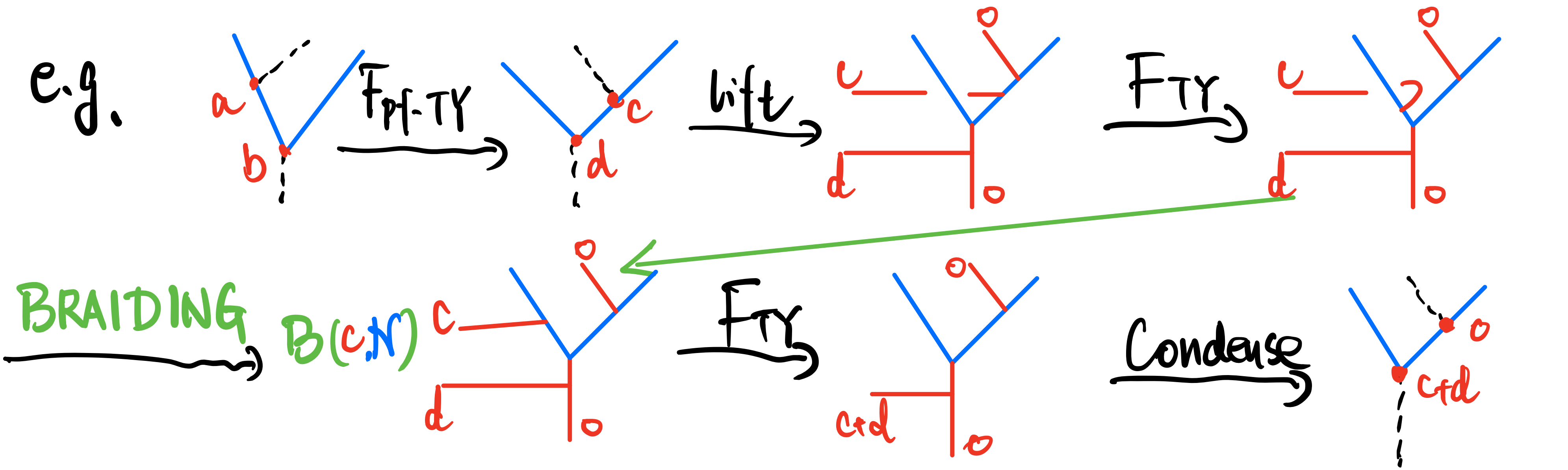


F — $N^2 \times N^2$ matrices

* Half-Braiding assign a map from

$$\{F\}_{TY^{t,\kappa}_{\mathbb{Z}_N}} \rightsquigarrow \{F\}_{pf\text{-}TY^{t,\kappa,B}_{\mathbb{Z}_N}}$$

e.g.



* $TY_{\mathbb{Z}_N}^{t,\kappa}$ + (half)-BRAIDING $\rightsquigarrow$ pf-$TY_{\mathbb{Z}_N}^{t,\kappa,B}$.

The triplet (t, κ, B) fully classify all pf-$TY_{\mathbb{Z}_N}$

* $TY^{t,\kappa}_{\mathbb{Z}_N}$ + (half)-BRAIDING $\rightsquigarrow$ pf-$TY^{t,\kappa,B}_{\mathbb{Z}_N}$.

The triplet (t, κ, B) fully classify all pf-$TY_{\mathbb{Z}_N}$

* In sum,
  - # t: $1 \le t < N$, $\gcd(t, N) = 1$
  - # κ: 2-choice, $\kappa = \pm 1$
  - # B:
    - 1-choice N ODD $B = (-1)^t \omega^{-1/2}$
    - 2-choice N EVEN $B = \pm \omega^{-1/2}$

e.g. Case of $N=2$, (Maj. Fermi-)

$t = 1$

$\kappa = 2$

$B = 2$

$\rightsquigarrow$ 4 sol.

— $\mathbb{Z}_2$-sym. Classification in Fermionic CFTs. ($\mathbb{Z}_2^F$)

[Chang, JC, Xu, 2022]

*. Construct $\mathbb{Z}_2^F$ from $\mathbb{Z}_4^B$ by gauging $\mathbb{Z}_2 \subset \mathbb{Z}_4^B$

$$\mathbb{Z}_4^B = \{0, \ 1, \ 2, \ 3\}.$$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4^B \longrightarrow \mathbb{Z}_2^F \longrightarrow 0$$

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conformal weight $h_a = \frac{k}{8} a^2 \pmod 1$		0	1	2	3	
	$k=1$	0	$1/8$	$1/2$	$1/8$	Anomalous $\mathbb{Z}_4^B$
	$k=2$	0	$1/4$	1	$1/4$	Non-anomalous $\mathbb{Z}_4^B$

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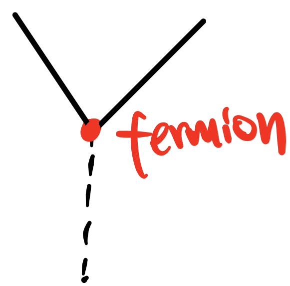
⇓ gauge/Condense $\mathbb{Z}_2$

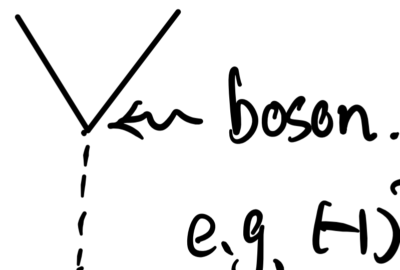
$k=1$

$\tilde{\mathbb{Z}}_2^m$



$k=2$

$\mathbb{Z}_2^m$



e.g. $(-1)^F$.

* Construct $\mathbb{Z}_2^F$ From Category by gauging $\mathbb{Z}_2 \subset \mathcal{C}$.

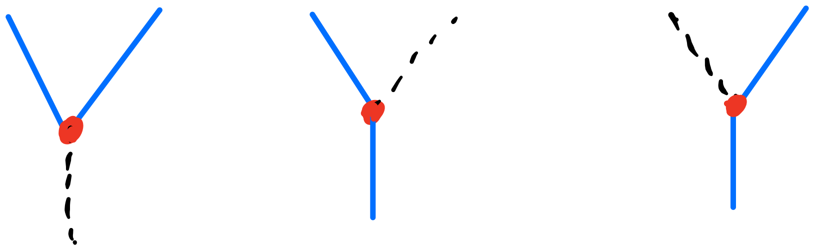
$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{C} \longrightarrow \mathbb{Z}_2^F \longrightarrow 0$$

* Construct $\mathbb{Z}_2^F$ from Category by gauging $\mathbb{Z}_2 \subset \mathcal{C}$.

$$0 \to \mathbb{Z}_2 \to \mathcal{C} \to \mathbb{Z}_2^F \to 0$$

$\rightsquigarrow$ $\mathcal{C}$ - Ising Cat $\{1, \eta, N\}$ with $h_\eta = 1/2$.

$\Big\downarrow$ gauge/condense $\mathbb{Z}_2$.



q-type $\mathbb{Z}_2^q$.

* Construct $\mathbb{Z}_2^F$ from Category by gauging $\mathbb{Z}_2 \subset \mathcal{C}$.

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{C} \longrightarrow \mathbb{Z}_2^F \longrightarrow 0$$

$\rightsquigarrow$ $\mathcal{C}$ – Ising Cat $\{ \mathbb{1}, \eta, N \}$ with $h_\eta = 1/2$.

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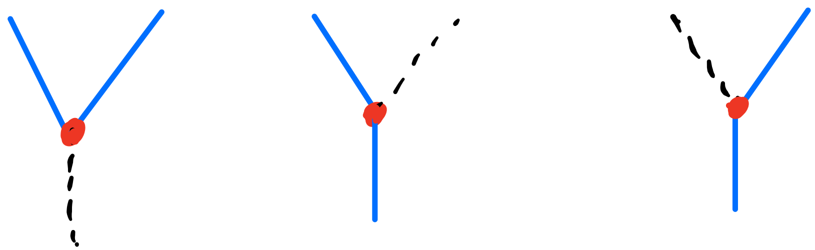

q-type $\mathbb{Z}_2^q$.

* Overall $\mathbb{Z}_2^F = \begin{cases} \mathbb{Z}_2^m & 2 \text{ sol} \\ \tilde{\mathbb{Z}}_2^m & 2 \text{ sol} \\ \mathbb{Z}_2^q & 4 \text{ sol.} \end{cases}$ $\rightsquigarrow \mathrm{Hom}\left(\Omega_3^{Spin}(B\mathbb{Z}_2), U(1)\right) \simeq \mathbb{Z}_8$

IV. Summary & Outlook.

— Propose $\mathbb{Z}_N$-para-Fusion CAT for TDLs in pf-CFTs

— Classify a class of universal TDLs in pf-CFTs.

— Many things to do in FUTURE.

- classification of TDLs in pf-CFTs
- 3D TFTs? $Z(\mathcal{C}_{PARA})$?
- enriched topological order. in CMT
- - - - - - -

THANK YOU!