Modular factorization of superconformal index

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1. Introduction

2. Review
   - Review of 2d $\mathrm{SL}(2,\mathbb{Z})$ modularity
   - The development of AdS_5 black holes

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Introduction

- Black hole entropy and temperature should be understood from quantum gravity point of view.

$$S = \frac{Ac^3}{4G\hbar}$$

- Microscopic states: Strominger and Vafa's work in 1996
- $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence: modularity relation
- How can one generalize these understandings (such as modularity) to higher dimensional AdS/CFT?

AdS$_3$/CFT$_2$

The first example of counting the microscopic states of black hole can be understood as AdS$_3$/CFT$_2$. The Bekenstein-Hawking entropy of BTZ black hole can be written as

$$S = \frac{A}{4G} = \pi\sqrt{\frac{\ell(\ell\mathcal{M}+\mathcal{J})}{2G}} + \pi\sqrt{\frac{\ell(\ell\mathcal{M}-\mathcal{J})}{2G}}$$

The AdS/CFT correspondence relates the conformal dimension to energy of black hole, and spin to the angular momentum

$$\mathcal{M} \leftrightarrow \Delta, \qquad \mathcal{J} \leftrightarrow s$$

We can also split the 2d CFT into the left moving modes and the right moving modes as

$$L_0 - \frac{c}{24} = \ell\mathcal{M} + \mathcal{J}, \qquad \bar{L}_0 - \frac{c}{24} = \ell\mathcal{M} - \mathcal{J}$$

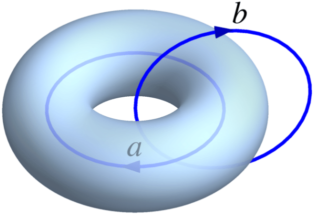
Then understanding the entropy reduces to understanding the Cardy formula from CFT$_2$

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{c}{6}\left(\bar{L}_0 - \frac{c}{24}\right)}$$

Modularity

The 2d partition function $Z[\tau, \bar{\tau}] = \mathrm{Tr}\, e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0}$ satisfies the SL$(2, \mathbb{Z})$ modularity (a powerful symmetry!)

$$Z_0\left[\tau\right] = Z_0\left[-\frac{1}{\tau}\right], \quad Z_0\left[\tau\right] = Z[\tau] e^{-2\pi i \tau \frac{c}{24}}$$



This indicates the asymptotic expansion of partition function near $\tau \to 0$ is

$$Z_0[\tau] \to e^{\frac{2\pi i c}{24\tau} + \frac{2\pi i c \tau}{24}} \times (1 + ...)$$

Cardy formula

The degeneracy of states with given energy is determined by inverse Legendre transformation

$$\rho(\Delta)=\int e^{-2\pi i\tau\Delta}e^{\frac{2\pi ic}{24\tau}+\frac{2\pi ic\tau}{24}}Z\left[-\frac{1}{\tau}\right]d\tau$$

Since $Z\left[-\frac{1}{\tau}\right]\to 1$, we can evaluate this integral by saddle point approximation

$$\tau=i\sqrt{\frac{c}{24\Delta}}$$

Then we get the Cardy formula

- This means in the high temperature, the universal behavior of $Z_0[\tau]$

$$\text{Casimir energy}+\mathbb{1}$$

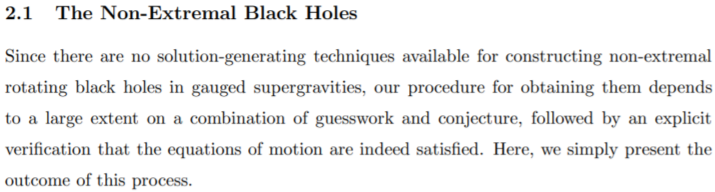
- The modularity originates from conformal symmetry

How to generalize this to higher dimensions?

Large AdS black hole

We consider black holes in AdS$_5 \times S^5$, which includes two angular momenta and three R-charges Q_1, Q_2, Q_3. Our first task is of course to find the large AdS$_5$ black hole solution. This is a rather difficult problem since Einstein gravity is a non-linear equation.

- (Chong, Cvetic, Lv, Pope): non-supersymmetric solutions but some equal charge/angular momentum



2.1 The Non-Extremal Black Holes

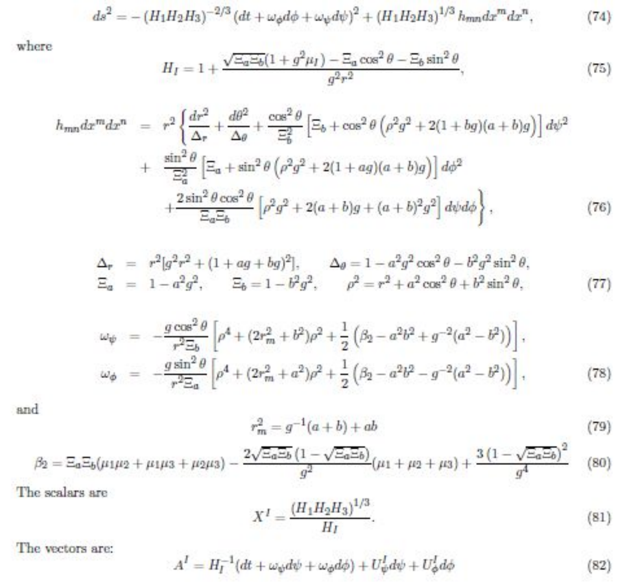
Since there are no solution-generating techniques available for constructing non-extremal rotating black holes in gauged supergravities, our procedure for obtaining them depends to a large extent on a combination of guesswork and conjecture, followed by an explicit verification that the equations of motion are indeed satisfied. Here, we simply present the outcome of this process.

- (Kunduri, Lucietti, Reall): only BPS black hole solutions.

$$E = J_1 + J_2 + Q_1 + Q_2 + Q_3$$

  The most general BPS black hole was found in 0601156.
- (Wu): the most general non-supersymmetric solution

Solution



$$ds^2 = -\left(H_1H_2H_3\right)^{-2/3}\left(dt + \omega_\phi d\phi + \omega_\psi d\psi\right)^2 + \left(H_1H_2H_3\right)^{1/3} h_{mn}dx^{\mathrm{m}}dx^{\mathrm{n}}, \tag{74}$$

where

$$H_I = 1 + \frac{\sqrt{\Xi_a\Xi_b}(1+g^2\mu_I) - \Xi_a\cos^2\theta - \Xi_b\sin^2\theta}{g^2r^2}, \tag{75}$$

$$\begin{aligned} h_{mn}dx^{\mathrm{m}}dx^{\mathrm{n}} &= r^2\Bigg\{\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} + \frac{\cos^2\theta}{\Xi_b^2}\left[\Xi_b + \cos^2\theta\left(\rho^2g^2 + 2(1+bg)(a+b)g\right)\right]d\psi^2 \\ &+ \frac{\sin^2\theta}{\Xi_a^2}\left[\Xi_a + \sin^2\theta\left(\rho^2g^2 + 2(1+ag)(a+b)g\right)\right]d\phi^2 \\ &+ \frac{2\sin^2\theta\cos^2\theta}{\Xi_a\Xi_b}\left[\rho^2g^2 + 2(a+b)g + (a+b)^2g^2\right]d\psi d\phi\Bigg\}, \end{aligned} \tag{76}$$

$$\begin{aligned} \Delta_r &= r^2[g^2r^2 + (1+ag+bg)^2], \qquad \Delta_\theta = 1 - a^2g^2\cos^2\theta - b^2g^2\sin^2\theta, \\ \Xi_a &= 1 - a^2g^2, \qquad \Xi_b = 1 - b^2g^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \end{aligned} \tag{77}$$

$$\begin{aligned} \omega_\psi &= -\frac{g\cos^2\theta}{r^2\Xi_b}\left[\rho^4 + (2r_m^2 + b^2)\rho^2 + \frac{1}{2}\left(\beta_2 - a^2b^2 + g^{-2}(a^2-b^2)\right)\right], \\ \omega_\phi &= -\frac{g\sin^2\theta}{r^2\Xi_a}\left[\rho^4 + (2r_m^2 + a^2)\rho^2 + \frac{1}{2}\left(\beta_2 - a^2b^2 - g^{-2}(a^2-b^2)\right)\right], \end{aligned} \tag{78}$$

and

$$r_m^2 = g^{-1}(a+b) + ab \tag{79}$$

$$\beta_2 = \Xi_a\Xi_b(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3) - \frac{2\sqrt{\Xi_a\Xi_b}\,(1-\sqrt{\Xi_a\Xi_b})}{g^2}(\mu_1+\mu_2+\mu_3) + \frac{3\,(1-\sqrt{\Xi_a\Xi_b})^2}{g^4} \tag{80}$$

The scalars are

$$X^I = \frac{(H_1H_2H_3)^{1/3}}{H_I}. \tag{81}$$

The vectors are:

$$A^I = H_I^{-1}(dt + \omega_\psi d\psi + \omega_\phi d\phi) + U_\psi^I d\psi + U_\phi^I d\phi \tag{82}$$

Large AdS black hole entropy

In the most general BPS black hole solution in AdS$_5 \times S^5$, the entropy is [Kunduri, Reall, Lucietti, 06]

$$S = \frac{A}{4G} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{N^2}{2}(J_1 + J_2)}$$

with the extremality condition

$$\left(Q_1Q_2 + Q_2Q_3 + Q_1Q_3 - \frac{N^2}{2}\left(J_1 + J_2\right)\right)\left(Q_1 + Q_2 + Q_3 + \frac{N^2}{2}\right)$$
$$= \frac{N^2}{2}J_1J_2 + Q_1Q_2Q_3$$

How to understand this entropy from the dual $\mathcal{N} = 4$ SYM calculation?

Fail trials

In 2005, Maldacena, Raju, Minwalla and Kinney studied the superconformal index, but they found the superconformal index in the large N limit scales as $\mathcal{O}(N^0)$ while black hole entropy scales as $\mathcal{O}(N^2)$. This mismatch was believed due to cancellation between fermionic and bosonic degrees of freedom.

Resolution

It was first realized by (Hosseini, Hristov, Zaffroni) that this entropy can be acquired by inverse Legendre transformation of the partition function

$$\ln Z = -i\pi \frac{N^2}{2} \frac{\phi_1\phi_2\phi_3}{\tau\sigma}$$

under the chemical potential constraint $\phi_1 + \phi_2 + \phi_3 - \tau - \sigma = 1$ which is equivalent to extremization of the entropy functional

$$\begin{aligned} S = &-i\pi \frac{N^2}{2} \frac{\phi_1\phi_2\phi_3}{\tau\sigma} - 2\pi i(\tau J_1 + \sigma J_2 + Q_1\phi_1 + Q_2\phi_2 + Q_3\phi_3) \\ &- \pi i\Lambda(\phi_1 + \phi_2 + \phi_3 - \tau - \sigma - 1) \end{aligned}$$

Breakthrough

- Chemical potentials are complex
- The leading order of partition function is very similar to the known supersymmetric Casimir energy

The chemical potential constraint

$\phi_1 + \phi_2 + \phi_3 - \tau - \sigma = 1$ is the conjugate version of BPS condition.

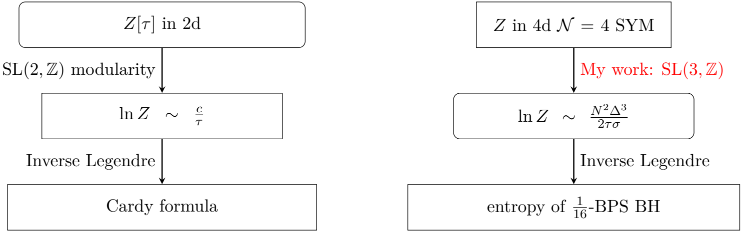
- Begin black hole thermodynamics data
- Supersymmetry

  $$E = Q_1 + Q_2 + Q_3 + J_a + J_b$$

  is not equivalent to extremality condition $r_+ = r_-$!
- Imposing supersymmetry condition first will force some parameters become complex, but return to be real for extremality condition to be satisfied.

$$q = a + b + ab - i(r_+^2 - \frac{ab}{1+a+b})$$

The analogy



$Z[\tau]$ in 2d

$\mathrm{SL}(2,\mathbb{Z})$ modularity

$\ln Z \sim \frac{c}{\tau}$

Inverse Legendre

Cardy formula

Z in 4d $\mathcal{N}=4$ SYM

My work: $\mathrm{SL}(3,\mathbb{Z})$

$\ln Z \sim \frac{N^2\Delta^3}{2\tau\sigma}$

Inverse Legendre

entropy of $\frac{1}{16}$-BPS BH

Our work is to find the analogue of $\mathrm{SL}(2,\mathbb{Z})$ modularity in 4d, which turns out to be $\mathrm{SL}(3,\mathbb{Z})$ modularity. Many researches in this topic recently: [Kim, Murthy, Choi, Milan, Benini, Nian, Zayas, Cassani, Cabo-Bizet, etc...]

Elliptic Gamma function

$$I = \mathrm{tr}(e^{-\beta E - Q_1\Delta_1 - Q_2\Delta_2 - Q_3\Delta_3 - J_1\tau - J_2\sigma})$$

The integral form of superconformal index is

$$I_N = \frac{\kappa_N}{N!}\prod_{k=1}^{N-1}\oint_{|x_k|=1}\frac{dx_k}{2\pi i x_k}\prod_{1\le i\ne j\le N}\frac{\prod_{a=1}^{3}\Gamma(x_{ij}f_a)}{\Gamma(x_{ij})}.$$

where the elliptic Gamma function is defined as

$$\Gamma(x) \equiv \Gamma(z;\tau,\sigma) = \prod_{m,n=0}^{\infty}\frac{1 - x^{-1}p^{m+1}q^{n+1}}{1 - xp^m q^n}$$

where $q = e^{2\pi i\tau}$, $p = e^{2\pi i\sigma}$, $x = e^{2\pi i z}$ The analogue function in 2d CFT is q-θ function

$$\theta(z;\tau) = \prod_{n=0}^{\infty}(1 - xq^n)(1 - x^{-1}q^{n+1})$$

Elliptic Gamma function

This was introduced by [Ruijsenaars, 97]. Some properties

$$\begin{aligned}
&\Gamma(z+\tau,\tau,\sigma)=\theta(z,\sigma)\Gamma(z,\tau,\sigma)\\
&\Gamma(z,\tau,\sigma)\Gamma(\tau+\sigma-z,\tau,\sigma)=1\\
&\Gamma(z,-\tau,\sigma)=\Gamma(\sigma-z,\tau,\sigma)\\
&\Gamma(z,-\tau,\sigma)=\frac{1}{\Gamma(z+\tau,\tau,\sigma)}\\
&\Gamma(z,\tau,\sigma)=\Gamma(z,\tau+\sigma,\sigma)\Gamma(z+\tau,\tau+\sigma,\tau)
\end{aligned}$$

Applications

- Baxter's formula of free energy for eight vertex model (Junya mentioned this model this morning)
- hypergeometric solutions of elliptic qKZB difference equations
- Partition function of chiral multiplet
- Show Seiberg duality by checking partition functions equal (Dan Xie's talk, [Dolan, Osborn, 08])

Modularity

The θ-function has the following SL$(2,\mathbb{Z})$ modularity transformation:

$$\theta\left(\frac{z}{\tau};-\frac{1}{\tau}\right)=e^{i\pi B(z,\tau)}\theta(z;\tau)\,,$$
$$B(z,\tau)=\frac{z^2}{\tau}+z\left(\frac{1}{\tau}-1\right)+\frac{1}{6}\left(\tau+\frac{1}{\tau}\right)-\frac{1}{2}\,.$$

$\tau\to 0$, $\theta(z/\tau,-1/\tau)\to 1$ approaches to 1.
The modularity satisfied by elliptic Gamma function is [Felder, Varchenko, 99]

$$\Gamma(z,\tau,\sigma)=e^{-i\pi Q(z,\tau,\sigma)}\Gamma\left(\frac{z}{\sigma},-\frac{1}{\sigma},\frac{\tau}{\sigma}\right)\Gamma\left(\frac{z}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\,,$$
$$Q(z;\tau,\sigma)=\frac{z^3}{3\tau\sigma}-\frac{\tau+\sigma-1}{2\tau\sigma}z^2+\frac{\tau^2+\sigma^2+3\tau\sigma-3\tau-3\sigma+1}{6\tau\sigma}z$$
$$+\frac{1}{12}(\tau+\sigma-1)(\tau^{-1}+\sigma^{-1}-1)$$

Cardy limits and entropy

$$\Gamma(z,\tau,\sigma) = e^{-i\pi Q}\Gamma\left(\frac{z}{\sigma}, -\frac{1}{\sigma}, \frac{\tau}{\sigma}\right)\Gamma\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\sigma}{\tau}\right)$$

In the Cardy limit $\tau, \sigma \to 0$, we can show

$$\Gamma(z,\tau,\sigma) \sim e^{-i\pi Q([z],\tau,\sigma)}$$

These will prove the

$$\ln Z \sim \frac{N^2}{2}\frac{\phi_1\phi_2\phi_3}{\tau\sigma}$$

Choi, Kim et.al; Milan, Benini; Murthy et.al 2018

Puzzle

What is the origin of this modular property? And anything like Farey tail?

- No modular symmetry
- Aim: give physical interpretations of the modular properties

Another motivation: Farey tail

In AdS$_3$/CFT$_2$,

$$\tau \to 0, \quad \text{BTZ black hole}$$

$$\tau \to \infty \quad \text{thermal AdS.}$$

We need to sum over all the saddle point in the partition function. [Dijkgraaf, et al, 2000]
These are called the SL$(2,\mathbb{Z})$ family of black holes. They are very crucial in understanding information paradox.

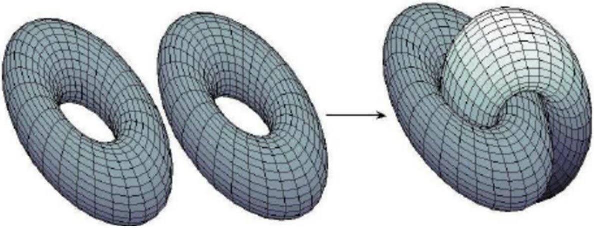
$$\mathcal{Z}_\chi(\beta,\omega) = -2\pi i \sum_{(c,d)=1, c\geq 0} \sum_{\mu=1}^{k} \sum_{4km-\mu^2<0} \tilde{c}_\mu\left(4km-\mu^2; \mathrm{Sym}^k(K3)\right)$$
$$(c\tau+d)^{-3} \exp\left[2\pi i (m - \frac{\mu^2}{4k}) \frac{a\tau+b}{c\tau+d}\right] \exp[-2\pi i k \frac{c\omega^2}{c\tau+d}] \Theta^+_{\mu,k}(\frac{\omega}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$$

Question

Can one do this in 4d SCFT?

Geometric background

The three dimensional manifold has the following topological operation:



Given two solid torus $D_2 \times S^1$. We can glue $(1,0)$ circle of one torus with $(0,1)$ circle of the other one. The gluing results in manifold S^3. Recall $\tau = x_2/x_1$.

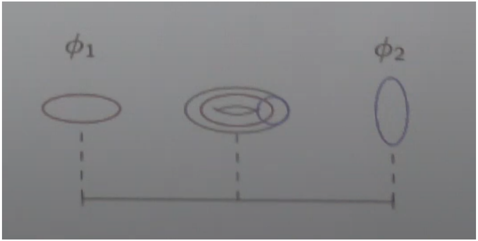
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = S \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The procedure is known as genus 1 Heegaard splitting.

$$M_3 \cong H_g \overset{f}{\sqcup} H_g'$$

Heegaard splitting

- $\Sigma_g = \partial H_g$. Then Σ_g and Σ'_g are identified orientation reversing diffeo f.
- f is classified by mapping class group $\mathsf{MCG}[\Sigma_g] = \mathsf{Diff}(\Sigma_g)/\mathsf{Diff}_0(\Sigma_g)$
- Example 1: $S^2 \times S^1$: $f = \mathbb{1}\mathcal{O}$
- Example 2: $S^3 \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1$, $f = S\mathcal{O}, S \in SL(2,\mathbb{Z})$.
  $H_1 = D_2 \times S^1$

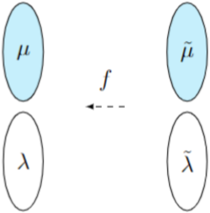


Lens geometry

For a lens geometry $L(p,q)$, the gluing is done by

$$f = \begin{pmatrix} s & -r \\ p & -q \end{pmatrix}$$

This description realizes $L(p,q)$ as a torus fibration over an interval with a $(1,0)$ cycle shrinking on one endpoint and a (q,p) cycle on the other



where $L(p,q) = \{(z_1, z_2) \in S^3 | (z_1, z_2) \sim (e^{\frac{2\pi i q}{p}} z_1, e^{-\frac{2\pi i}{p}} z_2)\}$. We denote $L(0,-1) = S^2 \times S^1$.

Redundancy

- λ and $\tilde{\lambda}$ are defined modulo $\mu, \tilde{\mu}$. This gives redundancy in f.
- $\Gamma_\infty : (\lambda, \mu) \to (\lambda + k\mu, \mu)$

Results in $L(p, q) = L(p, p + q)$

- All the $L(p, q)$ for different q have the same homotopy group.
- The lens geometry are equivalent iff (Reidemeister, 1935)

$$q \equiv \pm q' \mod p, \qquad qq' \equiv \pm 1 \mod p$$

- Reidemeister torsion

$$\tau_\xi \sim (1 - \xi^q)(1 - \xi), \qquad \xi^p = 1, \qquad \xi \neq 1$$

Redundancies of various definitions

The Heegaard splitting definitions are redundant also

Four dimensional manifold

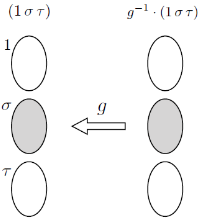
In four dimensions, we are using two three solid tori $D_2 \times T^2$ to do Heegaard splitting gluing. There are three S^1, which results in SL$(3, \mathbb{Z})$ modularity. We then use S_{23} to glue two solid torus to acquire $S^3 \times S^1$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = S_{23} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

which is written as

$$\mathcal{M}_g = (D_2 \times T^2) \times_g (D_2 \times T^2)$$

When $g = \mathbb{1}\mathcal{O}$, $M_g = S^2 \times T^2$. The moduli is $\tau = x_2/x_1, \sigma = x_3/x_1$.



Lens topology in 4d

We want to consider the $L(p,q)\times S^1$ topology. The matrix

$$h=\begin{pmatrix} * & 0 & * \\ * & 1 & * \\ * & 0 & * \end{pmatrix} \qquad \text{for} \quad h\in H\subset SL(3,\mathbb{Z})\,.$$

preserves the contractible cycle μ. This group is $SL(2,Z)_{\lambda\lambda'}\ltimes\mathbb{Z}^2$. Then two matrix $M_f\cong M_{f'}$ iff

$$f'=hf\tilde{h}^{-1}, \qquad h,\tilde{h}\in H$$

The crucial point is for any $g\in SL(3,Z)$, there exist $h,\tilde{h}$ such that

$$f_{p,q}=hg\tilde{h}^{-1}, \qquad f_{p,q}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & s & -r \\ 0 & p & -q \end{pmatrix}$$

This indicates $M_f\cong L(p,q)\times S^1$.

$$\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}})\cong M_f(\boldsymbol{\rho},\tilde{\boldsymbol{\rho}})\equiv D_2\times T^2(\boldsymbol{\rho})\overset{f}{\sqcup}D_2\times T^2(\tilde{\boldsymbol{\rho}})$$

Ambiguity

Claim: A given Hopf surface can be split into any such $M_{f'}(\boldsymbol{\rho})$

$$\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}}) = D_2 \times T^2(h\boldsymbol{\rho}) \overset{hf_{p,q}\tilde{h}^{-1}}{\sqcup} D_2 \times T^2(\tilde{h}f_{p,q}^{-1}\boldsymbol{\rho})$$

$H \times H$ action reflects the ambiguity in Heegaard splitting.

Physical theories

It is shown that SUSY partition function of 4d $\mathcal{N} = 1$ with $U(1)_R$ symmetry formulated on $\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}})$ only depends on $\boldsymbol{\rho}$. [Closset et.al, 13]

$$\mathcal{I}_{(1,0)}(\hat{\boldsymbol{\rho}}) = \mathrm{tr}_{\mathcal{H}}(-1)^F \hat{p}^{j_1+j_2-\frac{r}{2}} \hat{q}^{j_1-j_2-\frac{r}{2}} \hat{x}_a^{q_a} e^{-\beta\delta}$$

Factorization

The index is factorizable (by Higgs branch or localization) [Nieri, Pasquetti, 15]

$$\mathcal{I}_{(p,q)}(\hat{\boldsymbol{\rho}}) = e^{-i\pi\mathcal{P}_f(\boldsymbol{\rho})} \sum_{\alpha} \mathcal{B}_S^{\alpha}(\boldsymbol{\rho})\mathcal{B}_S^{\alpha}(f_{p,q}^{-1}\boldsymbol{\rho})$$

Consider the $\mathcal{N} = 1$ chiral multiplet defined on $D_2 \times T^2$. Using localization method (on $D_2 \times T^2$) of chiral multiplet:

$$\mathcal{B}_S^{\alpha}(\boldsymbol{\rho}) = \mathcal{B}^{\alpha}(S_{13}\boldsymbol{\rho}) = \Gamma\left(\frac{z}{\sigma}, \frac{\tau}{\sigma}, -\frac{1}{\sigma}\right)$$

These are called the holomorphic block. The factorization is called holomorphic factorization, reflecting Heegaard splitting:

$$\mathcal{M}_{(p,q)}(\hat{\boldsymbol{\rho}}) = D_2 \times T^2(S_{13}\boldsymbol{\rho}) \overset{S_{13}f_{p,q}S_{13}^{-1}}{\sqcup} D_2 \times T^2(S_{13}f_{p,q}^{-1}\boldsymbol{\rho})$$

Similarly the partition function on $S^3 \times S^1$ can also be shown as $\Gamma(z, \tau, \sigma)$.

$\mathrm{SL}(3,\mathbb{Z})$ modularity of elliptic Gamma function

$$\Gamma(z,\tau,\sigma) = e^{-i\pi Q(z,\tau,\sigma)}\Gamma\left(\frac{z}{\sigma},\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)\Gamma\left(\frac{z}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)$$

is equivalent to

$$\mathcal{I}^{\chi_0}_{(1,0)}(\boldsymbol{\rho}) \sim \mathcal{B}^{\chi_0}_S(\boldsymbol{\rho})\mathcal{B}^{\chi_0}_S(f^{-1}_{(1,0)}\boldsymbol{\rho})$$

However, nothing special about S_{13}! There are a lot of identities of elliptic Gamma functions:

$$\begin{aligned} Z_{\mathcal{O}}(\boldsymbol{\rho}) &= \frac{1}{\theta(z;\sigma)} = \Gamma(z;\tau,\sigma)\Gamma(z;-\tau,\sigma) \\ \Gamma(z,\tau,\sigma) &= \Gamma(z,\tau-\sigma,\sigma)\Gamma(z,\sigma-\tau,\tau) \end{aligned} \tag{1}$$

Especially, if $f' = hf_{p,q}\tilde{h}^{-1}$ and $\boldsymbol{\rho}' = h\boldsymbol{\rho}$,

$$\mathcal{Z}_{f'}(\boldsymbol{\rho}') = \mathcal{Z}_{f_{p,q}}(\boldsymbol{\rho})$$

This is rather non-trivial at the level of factorization formula! Physically this is due to the fact that partition function is independent of boundary conditions on $\mathcal{B}^\alpha$

Examples of different boundary conditions

For Dirichlet and Robin boundary conditions, the blocks

$$\mathcal{B}^{\alpha}(\boldsymbol{\rho}) = Z_{\partial}^{\alpha}(z,\tau)\mathcal{C}^{\alpha}(\boldsymbol{\rho})$$

In terms of elliptic Gamma functions

$$\Gamma(z+\sigma,\tau,\sigma) = \theta(z,\tau)\Gamma(z,\tau,\sigma)$$

In terms of modularity

$$\begin{aligned}\Gamma(z,\tau,\sigma) &= e^{-\pi i Q(z,\tau,\sigma)}\Gamma\left(\frac{z}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{z}{\sigma},-\frac{1}{\sigma},\frac{\tau}{\sigma}\right)\\ &= e^{-\pi i Q(z-1,\tau,\sigma)}\Gamma\left(\frac{z-1}{\tau},-\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{z-1}{\sigma},-\frac{1}{\sigma},\frac{\tau}{\sigma}\right)\end{aligned}$$

Modular factorization proposal

For different boundary conditions of the blocks

$$\mathcal{B}_h^\alpha(\boldsymbol{\rho})\mathcal{B}_{\tilde{h}}^\alpha(f^{-1}\boldsymbol{\rho}) \cong \mathcal{C}_h^\alpha(\boldsymbol{\rho})\mathcal{C}_{\tilde{h}}^\alpha(f^{-1}\boldsymbol{\rho})$$

where the equality is up to phase polynomial.

$$e^{-i\pi\mathcal{P}_f^{\mathbf{m}}(\boldsymbol{\rho})}\mathcal{B}^\alpha(h\boldsymbol{\rho})\mathcal{B}^\alpha(\tilde{h}f_{(p,q)}^{-1}\boldsymbol{\rho}) = e^{-i\pi\mathcal{P}_f^{\mathbf{1}}(\boldsymbol{\rho})}\mathcal{B}^\alpha(\boldsymbol{\rho})\mathcal{B}^\alpha(f_{(p,q)}^{-1}\boldsymbol{\rho})\,,$$

where $h, \tilde{h} \in SL(2,\mathbb{Z})_{13} \ltimes \mathbb{Z}^2$. The most general solution

$$h = \begin{pmatrix} n & 0 & m \\ -cl+\kappa n & 1 & -ck+\kappa m \\ l & 0 & k \end{pmatrix}\,, \quad \tilde{h} = \begin{pmatrix} \mp n + pc & 0 & m \\ \mp c\tilde{l} + \tilde{\kappa}(\mp n + pc) & 1 & \mp c\tilde{k} + \tilde{\kappa} m \\ \tilde{l} & 0 & \tilde{k} \end{pmatrix}$$

The modular factorization proposal then claims the lens indices

$$\mathcal{I}_{(p,q)}(\hat{\boldsymbol{\rho}}) = e^{i\pi\mathcal{P}}\sum_\alpha \mathcal{B}_h^\alpha(\boldsymbol{\rho})\mathcal{B}_{\tilde{h}}^\alpha(f_{p,q}^{-1}\boldsymbol{\rho})$$

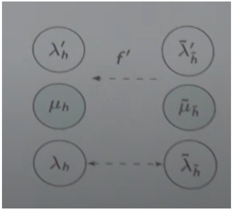
The $h, \tilde{h}$ are subgroups of $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ in general, becoming the full $SL(2,\mathbb{Z})$ in some transformation of $L(p,1)\times S^1$ or $S^3 \times S^1$.

Factorization condition

Results: any Heegaard splitting $M_{f'}(h\boldsymbol{\rho})$ allowed iff

$$f' = h f_{p,q} \tilde{h}^{-1} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \in F\mathcal{O}$$

Geometrically, F fixes a non-contractible cycle $\lambda_h = \tilde{\lambda}_{\tilde{h}}$



which is identified with S^1_β. Therefore $(h, \tilde{h})$ fixes the ambiguities of embedding S^1_β inside $D_2 \times T^2$ for fixed $f_{p,q}$.

General factorization of $S^3 \times S^1$ partition function

We consider the partition function $Z_{S_{23}\mathcal{O}}(\boldsymbol{\rho})$

$$\Gamma(z;\tau,\sigma) = e^{-i\pi Q_{\mathbf{m}}(z;\tau,\sigma)}\Gamma\left(\tfrac{z}{m\sigma+n};\tfrac{\tau-\tilde{n}(k\sigma+l)}{m\sigma+n},\tfrac{k\sigma+l}{m\sigma+n}\right)\Gamma\left(\tfrac{z}{m\tau+\tilde{n}};\tfrac{\sigma-n(\tilde{k}\tau+\tilde{l})}{m\tau+\tilde{n}},\tfrac{\tilde{k}\tau+\tilde{l}}{m\tau+\tilde{n}}\right)$$

where

$$Q_{\mathbf{m}}(z;\tau,\sigma) = \tfrac{1}{m}Q(mz;m\tau+\tilde{n},m\sigma+n) + f_{\mathbf{m}}$$

The $f_{\mathbf{m}} = 2\sigma_1(n,\tilde{n},1;m)$ is generalized Dedekind Sum for

$$\sigma_t(n_1,n_2\cdots,n_r;m) = \frac{1}{m}\sum_{\xi^m=1\neq\xi}\frac{\xi^t}{(\xi^{n_1}-1)\cdots(\xi^{n_r}-1)}$$

θ function: the partition function of $S^2 \times T^2$

$$\theta\left(\frac{z}{m\tau+n};\frac{k\tau+l}{m\tau+n}\right) = e^{i\pi B_2^{\mathbf{m}}(z;\tau)}\theta(z;\tau)\,, \quad \mathbf{m}=(m,n)$$

and $B_2^{\mathbf{m}}(z,\tau) = \frac{1}{m}B_2(mz,m\tau+n) + 2\sigma_0(n,m)$

Physical effect

The structure indicates existence of other gravitational saddle points whose entropy is of order

$$S_m = \frac{1}{m} S_1$$

The gravitational solutions are subleading then.

Example: Lens geometry $L(p,q)$

$$\Delta_t \equiv g_{(p,q)} = S_{23} \prod_{i=1}^{t} \left(T_{23}^{-e_i} S_{23}\right) \,, \qquad e_i \geq 2$$

and

$$\Delta_i = S_{23} \prod_{j=1}^{i} \left(T_{23}^{-e_j} S_{23}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -s_i & -r_i \\ 0 & -p_i & -q_i \end{pmatrix}$$

$$\begin{aligned} p_i &= e_i p_{i-1} - p_{i-2}\,, \qquad & q_i &= p_{i-1}\,, \\ s_i &= e_i s_{i-1} - s_{i-2}\,, \qquad & r_i &= s_{i-1}\,, \\ p_0 = 1\,, \quad p_1 &= e_1\,, \quad s_0 = 0\,, \quad s_1 = 1\,. \end{aligned}$$

The p_i, q_i satisfy the continued fraction expansion [Popescu-Pampu, 05]

$$\frac{p_i}{q_i} = [e_i, e_{i-1}, \ldots, e_1]^- \equiv e_i - \frac{1}{e_{i-1} - \frac{1}{\cdots - \frac{1}{e_1}}}$$

Chiral multiplet

$$
\begin{aligned}
Z_{g_{(p,q)}\mathcal{O}}(\boldsymbol{\rho}) =& \prod_{i=0}^{t-1} \Gamma(z + p_{i-1}\tau - s_{i-1}\sigma; p_i\tau - s_i\sigma\,, p_{i-1}\tau - s_{i-1}\sigma) \\
& \times \Gamma(z; p_t\tau - s_t\sigma\,, p_{t-1}\tau - s_{t-1}\sigma) \\
=& \Gamma(z;\tau,\sigma) \prod_{i=1}^{t} \Gamma(z + p_i\tau - s_i\sigma; p_i\tau - s_i\sigma\,, p_{i-1}\tau - s_{i-1}\sigma)\,,
\end{aligned}
$$

Invariance

The lens index is invariant under large diffeomorphism of Hopf surface.

$$q \leftrightarrow s, \qquad \left(\tau - \frac{s}{p}\sigma\right) \leftrightarrow \frac{\sigma}{p}$$

Fatorization of Lens indices

$$
\begin{aligned}
\prod_{i=0}^{t} \Gamma(z+\sigma_i, \tau_i, \sigma_i) &= \prod_{i=0}^{t} e^{-i\pi Q'_i} \frac{\Gamma_{i+}}{\Gamma_{i-}} \\
&= \left(\prod_{i=0}^{t} e^{-i\pi Q'_i} \right) \frac{\Gamma_{0+}}{\Gamma_{0-}} \frac{\Gamma_{1+}}{\Gamma_{1-}} \cdots \frac{\Gamma_{t+}}{\Gamma_{t-}} \\
&= \left(\prod_{i=0}^{t} e^{-i\pi Q'_i} \right) \frac{\Gamma_{0+}}{\Gamma_{t-}}
\end{aligned}
$$

This results in

$$
\begin{aligned}
& Z_{g_{(p,q)}} \mathcal{O}(\boldsymbol{\rho}) \\
= \quad & e^{-i\pi \tilde{P}^{\mathbf{m}}_{g_{(p,q)}}(z,\tau,\sigma)} \Gamma \left(\tfrac{z}{m\sigma+n_1} ; \tfrac{\tau - c(k_1\sigma + l_1)}{m\sigma+n_1}, \tfrac{k_1\sigma+l_1}{m\sigma+n_1} \right) \\
& \times \Gamma \left(\tfrac{z}{m(p\tau - s\sigma)+\tilde{n}_{t+1}} ; \tfrac{q\tau - r\sigma - n_{t+1}(\tilde{k}_{t+1}(p\tau - s\sigma) + \tilde{l}_{t+1})}{m(p\tau - s\sigma)+\tilde{n}_{t+1}}, \tfrac{\tilde{k}_{t+1}(p\tau - s\sigma)+\tilde{l}_{t+1}}{m(p\tau - s\sigma)+\tilde{n}_{t+1}} \right) ,
\end{aligned}
$$

Phase polynomial

$$\tilde{P}^{\mathbf{m}}_{g_{(p,q)}}(\boldsymbol{\rho}) = \frac{1}{mp} Q\left(mz, \frac{m(p\tau - s\sigma) + \tilde{n}_{t+1}}{p}, \frac{m\sigma + n_1}{p}\right) + \delta\tilde{P}^{\mathbf{m}}_{g_{(p,q)}}(\boldsymbol{\rho})$$

$$\delta\tilde{P}^{\mathbf{m}}_{g_{(p,q)}}(\boldsymbol{\rho}) = \frac{(\eta_t + 3)p - 3}{6p} z - \frac{(p^2 - 1)(p\tau - s\sigma + \sigma)}{12p^2} + f_{\mathbf{m};(p,q)} \,,$$

Here $\eta_t = 12s(s,p)$ is the Dedekind Sum: [Stanislav, Sinai , Wang, 11]: Heegaard Floer correction terms..

Generalizations

$$\mathcal{P}^{(p,q)}(\vec{Z}; \hat{x}_i) \equiv \frac{1}{3p\hat{x}_1\hat{x}_2\hat{x}_3} \Big(k_{abc}Z_aZ_bZ_c + 3k_{abR}Z_aZ_bX + 3k_{aRR}Z_aX^2 \\ -k_aZ_a\tilde{X}^{(p,q)} + k_{RRR}X^3 - k_RX\tilde{X}^{(p,1)}\Big)$$

$$\tilde{X}^{(p,q)} = \frac{1}{4}(\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 - 2(p\eta_t + 3p - 3)\hat{x}_2\hat{x}_3), \quad X \equiv \frac{1}{2}\sum_{i=1}^{3}\hat{x}_i$$

Comparison between phase polynomial

Compare [Jejjala, Leuven, YL, Li, 22] with early results [Nieri, Pasquetti, 15]

$$\begin{aligned}\mathcal{P}^{(p,q)}(\vec{Z};\hat{x}_i) \equiv& \frac{1}{3p\hat{x}_1\hat{x}_2\hat{x}_3}\Big(k_{abc}Z_aZ_bZ_c+3k_{abR}Z_aZ_bX+3k_{aRR}Z_aX^2\\ &-k_aZ_a\tilde{X}^{(p,q)}+k_{RRR}X^3-k_RX\tilde{X}^{(p,1)}\Big)\\ \mathcal{P}^{(p,1)}(\vec{Z};\hat{x}_i) \equiv& \frac{1}{3p\hat{x}_1\hat{x}_2\hat{x}_3}\Big(k_{abc}Z_aZ_bZ_c+3k_{abR}Z_aZ_bX+3k_{aRR}Z_aX^2\\ &-k_aZ_a\tilde{X}^{(p,1)}+k_{RRR}X^3-k_RX\tilde{X}^{(p,1)}\Big)\\ \mathcal{P}_{S^3}(\vec{Z};\hat{x}_i) \equiv& \frac{1}{3\hat{x}_1\hat{x}_2\hat{x}_3}\Big(k_{abc}Z_aZ_bZ_c+3k_{abR}Z_aZ_bX+3k_{aRR}Z_aX^2\\ &-k_aZ_a\tilde{X}+k_{RRR}X^3-k_RX\tilde{X}\Big)\end{aligned}$$

where (from Ggg, Fgg, Rgg terms)

$$\tilde{X}^{(p,q)}=\frac{1}{4}(\hat{x}_1^2+\hat{x}_2^2+\hat{x}_3^2-2(p\eta_t+3p-3)\hat{x}_2\hat{x}_3),\quad X\equiv\frac{1}{2}\sum_{i=1}^{3}\hat{x}_i$$

Partition function relations

As application of modular factorization conjecture, we then prove the degree 1 automorphic form as the modularity. [Jejjala, Leuven, YL, Li, 22]

The partition function in 4d depends on the group element we choose. It has been shown they satisfy the following

$$Z_{g_1 g_2}(\vec{\tau}) = e^{i\phi_{g_1,g_2}} Z_{g_1}(\vec{\tau}) \cdot Z_{g_2}(g_1^{-1}\vec{\tau})$$

conjectured by [Gadde, 2000]
Recall the 2d partition function satisfies

$$Z(z,\tau) = e^{i\phi_g} Z(g^{-1} \cdot (z,\tau))$$

They are distinguished as automorphic form of degree 0 and 1.

Co-chain

Focus on the k-cochain group $C^k(G,A)$, where $A = N, M, N/M$. The group $C^k(G,A)$ consists of k-cocycles $\xi : G^k \to A$ such that $\xi_{g_1,\dots,g_k} = 1$ if $g_j = 1$ for some j. Furthermore, one defines $C^0(G,A) = A$. To construct the relevant cohomology groups, we now define a coboundary operator $\delta = \delta_k : C^k(G,A) \to C^{k+1}(G,A)$ via:

$$(\delta\xi)_{g_1,\dots,g_{k+1}}(\boldsymbol{\rho}) = \xi_{g_1,\dots,g_k}(\boldsymbol{\rho}) \left(\xi_{g_2,\dots,g_{k+1}}(g_1^{-1}\boldsymbol{\rho}) \prod_{j=1}^{k} \xi_{g_1,\dots,g_j g_{j+1},\dots,g_{k+1}}(\boldsymbol{\rho})^{(-1)^j} \right)^{(-1)^{k+1}}$$

So $\delta^2 = 1$ as one can verify.

Furthermore, for $k = 0$ one defines δ on $\chi \in C^0(G, A)$ as:

$$(\delta\chi)_g(\boldsymbol{\rho}) = \frac{\chi(\boldsymbol{\rho})}{\chi(g^{-1}\boldsymbol{\rho})}$$

The coboundary operator allows us to define cohomology in the usual way:

$$H^k(G, A) = \frac{\ker \delta_k}{\text{im } \delta_{k-1}}\,,\ k \geq 1\,, \quad H^0(G, A) = \ker \delta_0$$

If $\chi \in C^0(G, N) = N$, A degree 0 automorphic form of type $\xi_g \in C^1(G, M)$ corresponds to such a function χ which obeys:

$$(\delta\chi)_g(\boldsymbol{\rho}) = \frac{\chi(\boldsymbol{\rho})}{\chi(g^{-1}\boldsymbol{\rho})} = \xi_g(\boldsymbol{\rho})\,,$$

Since ξ_g is taking value in M, we notice that this equation abstracts the property associated to degree 0 Jacobi forms when $G = \mathcal{J}$. It also follows that such χ can be thought of as elements in $H^0(\mathcal{J}, N/M)$, since they are annihilated by δ modulo M.

Degree 1

Having set up the general framework, let us now increase the rank of the cohomology by one, and consider the action of δ_*. We take a one-cocycle $X_g \in C^1(G, N)$. Given the above, it follows that if $[X_g] \in H^1(G, N/M)$, it should satisfy:

$$\delta(X_{g_1}(\boldsymbol{\rho}))_{g_2} = \frac{X_{g_1}(\boldsymbol{\rho}) X_{g_2}(g_1^{-1}\boldsymbol{\rho})}{X_{g_1 g_2}(\boldsymbol{\rho})} = \xi_{g_1,g_2}(\boldsymbol{\rho})\,,$$

where $[\xi_{g_1,g_2}] \in H^2(G, M)$. It will be important in the following that for $H^1(G, N/M)$, there is a notion of trivializable or “exact” elements. Indeed, such classes can be written as:

$$[X_g] = [(\delta B)_g] = \left[\frac{B(\boldsymbol{\rho})}{B(g^{-1}\boldsymbol{\rho})}\right]\,, \tag{2}$$

with $B \in C^0(G, N)$.

Trivialization subgroup

$$\Gamma(z+\sigma;\tau,\sigma) = e^{-i\pi Q(z+\sigma;\tau,\sigma)} \frac{\Gamma\left(\frac{z}{\sigma};\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)}{\Gamma\left(\frac{z}{\tau};-\frac{\sigma}{\tau},-\frac{1}{\tau}\right)}\,. \tag{3}$$

Then, the equation can be written as follows:

$$X_{S_{23}}(\boldsymbol{\rho}) \cong \frac{B^S(\boldsymbol{\rho})}{B^S(S_{23}^{-1}\boldsymbol{\rho})}\,, \tag{4}$$

where we have defined the function $B^S(\boldsymbol{\rho})$:

$$B^S(\boldsymbol{\rho}) \equiv B(S_{13}\boldsymbol{\rho}) = \Gamma\left(\frac{z}{\sigma};\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)\,, \qquad B(\boldsymbol{\rho}) = \Gamma(z;\tau,\sigma)\,. \tag{5}$$

g sits in a subgroup of modular group SL$(3,\mathbb{Z}) \ltimes \mathbb{Z}^3$

$$F_S \equiv SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \qquad \text{with} \quad \begin{aligned} SL(2,\mathbb{Z}) &= \{S_{23}, T_{23}\}\,, \\ \mathbb{Z}^2 &= \{T_{12}, T_{13}\}\,. \end{aligned}$$

Other dimension

Dimension	Modular group	Automorphic form	Geometry
2	SL$(2, Z)$	0	Swapping $S^1 \times S^1$
3	SL$(2, Z)$	1	Heegaard splitting
4	SL$(3, Z)$	1	Heegaard splitting
5	SL$(3, Z)$	2	Trisection?
6	SL$(4, Z)$	2	Trisection?

See [Gukov, 17]

Summary and future works

- Ambiguity in Heegaard splitting leads to modular factorization, reflecting as an action in $H \times H$
- This is also crucially related to how S^1_β embedded in the $L(p,q)$.
- We work out the factorization of lens indices and compute the anomaly polynomial.

Future directions

- Modular bootstrap
- More identities and applications of elliptic Gamma functions?
- SL$(3,\mathbb{Z})$ Farey tail summation?
- AdS Black lens in gravitational theory
- Schur limit - Quantum modular forms [Leuven, 24] See also Yiwen's and Yanbin's paper for related topics