

2d CFTs, Borcherds products and hyperbolization of affine Lie algebras

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An interesting coincidence

8 special affine Lie algebras appeared in math and physics around the same time

Corollary (Corollary 7.6). *The following infinite series of pure theta blocks of q -order 1 satisfy the theta block conjecture.*

weight	root system	theta block
2	A_4	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_c\vartheta_d\vartheta_{a+b}\vartheta_{b+c}\vartheta_{c+d}\vartheta_{a+b+c}\vartheta_{b+c+d}\vartheta_{a+b+c+d}$
	$A_1 \oplus B_3$	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_{b+c}\vartheta_{b+2c+d}\vartheta_{b+c+d}\vartheta_{b+c+2d}\vartheta_{c+d}\vartheta_{c+2d}\vartheta_{c+d}$
	$A_1 \oplus C_3$	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_{2b+2c+d}\vartheta_{b+c}\vartheta_{b+2c+d}\vartheta_{b+c+d}\vartheta_{c}\vartheta_{2c+d}\vartheta_{c+d}\vartheta_d$
	$B_2 \oplus G_2$	$\eta^{-6}\vartheta_a\vartheta_{a+b}\vartheta_{a+2b}\vartheta_b\vartheta_c\vartheta_{3c+d}\vartheta_{3c+2d}\vartheta_{2c+d}\vartheta_{c+d}\vartheta_d$
3	$3A_2$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_1+b_1}\vartheta_{b_1}\vartheta_{a_2+b_2}\vartheta_{b_2}\vartheta_{a_3}\vartheta_{a_3+b_3}\vartheta_{b_3}$
	$3A_1 \oplus A_3$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_6}\vartheta_{a_4+a_5}\vartheta_{a_5+a_6}\vartheta_{a_4+a_5+a_6}$
	$2A_1 \oplus A_2 \oplus B_2$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_3+a_4}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_5+a_6}\vartheta_{a_6}$
4	$8A_1$	$\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_6}\vartheta_{a_7}\vartheta_{a_8}$

Figure: Dittmann, Wang, *Theta blocks related to root systems*, 2006.12967.

- F_{24} : This is a theory of 24 free chiral fermions. One can build an $\mathcal{N} = 1$ superconformal structure by taking a linear combination of cubic Fermi terms, and the allowed choices are classified by semisimple Lie algebras of dimension 24. Each of these generates an affine Kac-Moody algebra, of which there are eight possibilities:

$$(\widehat{su}(2)_2)^{\oplus 8}, \quad (\widehat{su}(3)_3)^{\oplus 3}, \quad \widehat{su}(4)_4 \oplus (\widehat{su}(2)_2)^{\oplus 3}, \quad \widehat{su}(5)_5, \quad \widehat{so}(5)_3 \oplus \widehat{g}_{2,4}, \\ \widehat{so}(5)_3 \oplus \widehat{su}(3)_3 \oplus (\widehat{su}(2)_2)^{\oplus 2}, \quad \widehat{so}(7)_5 \oplus \widehat{su}(2)_2, \quad \widehat{sp}(6)_4 \oplus \widehat{su}(2)_2,$$

Figure: Harrison, Paquette, Persson, Volpato, *Fun with F_{24}* , 2009.14710.

Main question

Elliptic Semi-simple Lie algebras – **Classical symmetries**

Parabolic Affine Kac-Moody algebras – **2d Wess-Zumino-Witten CFTs**

Hyperbolic Borcherds-Kac-Moody algebras – **Algebra of BPS states**
(**Harvey-Moore 95,96**)

Question (**Feingold-Frenkel 83, Gritsenko 12, Gritsenko-Wang 19,20...**)

What kinds of affine Kac-Moody algebras allow hyperbolization?

Roughly speaking, this is to classify the reflective Borcherds products $\Phi(\omega, \mathfrak{g}, \tau)$ of singular weight $\frac{1}{2}\text{rk}(L)$ on $2U \oplus L$, where U is hyperbolic plane and L is a positive definite lattice.

Main results

We give a complete classification of hyperbolization of affine Kac-Moody algebras

Main theorem (KS-Wang-Williams 23)

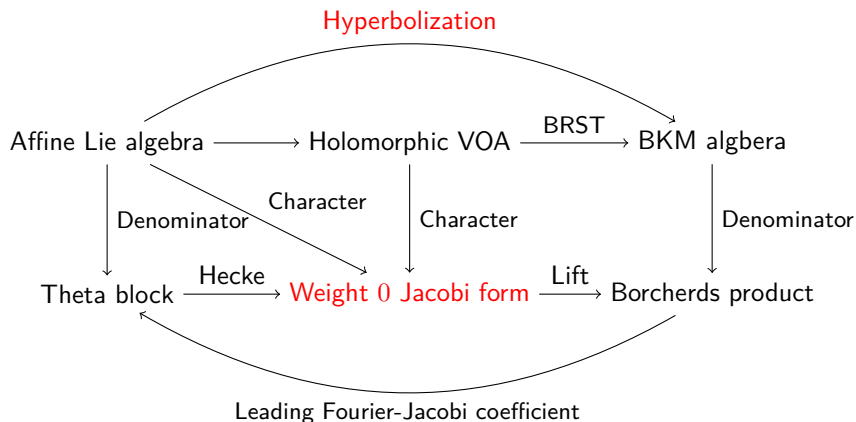
There are precisely 81 such affine Kac-Moody algebras:

- 1 69 cases associated to Schellenkens' list of $c = 24$ holomorphic CFT/VOAs
- 2 8 cases associated to the $c = 12$ holomorphic SCFT/self-dual SVOAs
- 3 4 exotic cases $A_{1,16}, A_{2,9}, A_{1,8}^2, A_{1,4}^4$ associated to exceptional modular invariants from nontrivial automorphism of fusion algebras

Remark

The class 1 is called **antisymmetric** and is also recently proved by (Driscoll-Spittler, Scheithauer, Wilhelm, 2312.02768) using different method. The class 2 and 3 are called **symmetric**.

Main idea



Outline

- 1 A simple example of Gritsenko-Nikulin $\Delta_{1/2}(Z)$
- 2 Reflective Borcherds products of singular weight
- 3 Hyperbolization of affine Kac-Moody algebras
- 4 69 CFT cases
- 5 8 SCFT cases
- 6 4 exotic cases
- 7 $osp(1|2n)$ and BKM algebras with odd real roots

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

Recall Jacobi symbol

$$\left(\begin{array}{c} -4 \\ m \end{array} \right) = \begin{cases} \pm 1, & m \equiv \pm 1 \pmod{4}, \\ 0, & m \equiv 0 \pmod{2}, \end{cases}$$

and Jacobi theta function

$$\theta_1(\tau, z) = \sum_{m \in \mathbb{Z}} \left(\begin{array}{c} -4 \\ m \end{array} \right) q^{m^2/8} r^{m/2}.$$

(Gritsenko-Nikulin 98) defined the Siegel theta constant

$$\Delta_{1/2}(Z) = \frac{1}{2} \sum_{m, n \in \mathbb{Z}} \left(\begin{array}{c} -4 \\ m \end{array} \right) \left(\begin{array}{c} -4 \\ n \end{array} \right) q^{m^2/8} r^{mn/2} s^{n^2/8},$$

and found this is a **weight-1/2** automorphic form for paramodular group Γ_4^+ .

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

Notably, $\Delta_{1/2}(Z)$ is a **reflective Borcherds product of singular weight!**

$$\Delta_{1/2}(Z) = \mathbf{B}(\phi) := q^A r^B s^C \prod_{\substack{n,l,m \in \mathbb{Z} \\ (n,l,m) > 0}} (1 - q^n r^l s^m)^{f(nm,l)},$$

where $f(-, -)$ are the Fourier coefficients of weight-0 Jacobi form ϕ :

$$\phi(\tau, z) := \frac{\theta_1(\tau, 3z)}{\theta_1(\tau, z)} = \sum_{n,l \in \mathbb{Z}} f(n, l) q^n r^l,$$

and

$$A = \frac{1}{24} \sum_l f(0, l) = \frac{1}{8}, \quad B = \frac{1}{2} \sum_{l>0} l f(0, l) = \frac{1}{2}, \quad C = \frac{1}{4} \sum_l l^2 f(0, l) = \frac{1}{8}.$$

Here \mathbf{B} is called the **Borcherds lift**.

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

- $\Delta_{1/2}(Z)$ is the denominator of a **Borcherds-Kac-Moody algebra** \mathfrak{g} with infinite dimensional Cartan matrix $a_{mn} = 2 - 4(m - n)^2$.
- This \mathfrak{g} can be regarded as the hyperbolization of affine Kac-Moody algebra $\mathfrak{g} = A_{1,16}$!
- Notice the weight-0 Jacobi form can be written as

$$\phi(\tau, z) = \frac{\theta_1(\tau, 3z)}{\theta_1(\tau, z)} = \chi_2^{\mathfrak{g}}(\tau, z) + \chi_{14}^{\mathfrak{g}}(\tau, z) - \chi_8^{\mathfrak{g}}(\tau, z).$$

- For $z \rightarrow 0$, this reduces to

$$3 = \chi_2^{\mathfrak{g}}(\tau) + \chi_{14}^{\mathfrak{g}}(\tau) - \chi_8^{\mathfrak{g}}(\tau),$$

which is a consequence of the **Macdonald identity** for $A_{1,2p^2-2}$ with $p = 3$:

$$p = \sum_{j=0}^{p-1} \chi_{2p^2-1-(4j+1)p}^{A_{1,2p^2-2}}(\tau).$$

- The same identity was used in (**Moore-Seiberg 88**) to construct the E_7 type modular invariant for \hat{A}_1 .

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

$\Delta_{1/2}(Z)$ is also a (trivial) **Gritsenko additive lift** of the A_1 theta block $\theta_{A_1} = \theta_1$:

$$\mathbf{B}(\phi) = \mathbf{G}(\theta_1)(\tau, z, \omega) = \sum_{m=1}^{\infty} \begin{pmatrix} -4 \\ m \end{pmatrix} \theta_1(\tau, mz) e^{2\pi i m^2 \omega / 8}.$$

In fact, $-\theta_1(\tau, 3z)$ can be defined as certain special Hecke image of the theta block $\theta_1(\tau, z)$ such that

$$\phi = -\frac{\theta_{A_1}|T}{\theta_{A_1}}.$$

These properties will persist for **all symmetric reflective Borcherds product of singular weight**.

Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

A closer look at the Fourier coefficients of the Jacobi form ϕ of $\Delta_{1/2}(Z)$:

$$\begin{aligned}\phi(\tau, z) &= \frac{\theta_1(\tau, 3z)}{\theta_1(\tau, z)} = \sum f(n, l)q^n x^l \\ &= (x^{\pm 1} + 1) + q(-x^{\pm 4} - x^{\pm 3} + x^{\pm 1} + 2) \\ &\quad + q^2(-x^{\pm 5} - 2x^{\pm 4} - 2x^{\pm 3} + 3x^{\pm 1} + 4) \\ &\quad + q^3(x^{\pm 7} - 2x^{\pm 5} - 4x^{\pm 4} - 4x^{\pm 3} + 5x^{\pm 1} + 8) \\ &\quad + q^4(x^{\pm 8} + x^{\pm 7} - 4x^{\pm 5} - 8x^{\pm 4} - 7x^{\pm 3} + 10x^{\pm 1} + 14) \\ &\quad + q^5(x^{\pm 9} + 2x^{\pm 8} + 3x^{\pm 7} - 7x^{\pm 5} - 14x^{\pm 4} - 13x^{\pm 3} + 16x^{\pm 1} + 24) + \dots\end{aligned}$$

The terms $f(n, l)q^n x^l$ with $16n - l^2 < 0$ are called **singular Fourier coefficients**. (Gritsenko-Nikulin 98) found **all singular Fourier coefficients is 1 and have $16n - l^2 = -1$** . This provides one of the necessary conditions for $\mathbf{B}(\phi)$ to be Borcherds product of singular weight.

Reflective Borcherds products of singular weight

We now move to the general theory on $2U \oplus L$.

Theorem (Gritsenko)

Let L be an even positive definite lattice of rank $\text{rk}(L)$. Let ϕ be a weakly holomorphic Jacobi form of trivial character with Fourier expansion

$$\phi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z}} \sum_{\ell \in L'} f(n, \ell) q^n \zeta^\ell.$$

Assume ϕ has *integral singular Fourier coefficients* (for $2n - (l, l) < 0$). Then the Borcherds product of ϕ is a meromorphic modular form of weight $f(0, 0)/2$ and can be expanded as

$$\mathbf{B}(\phi)(Z) = q^A \zeta^{\vec{B}} \xi^C \prod_{\substack{n, m \in \mathbb{Z}, \ell \in L' \\ (n, \ell, m) > 0}} \left(1 - q^n \zeta^\ell \xi^m\right)^{f(nm, \ell)},$$

where $q = \exp(2\pi i \tau)$, $\zeta^\ell = \exp(2\pi i(\ell, \mathfrak{z}))$, $\xi = \exp(2\pi i \omega)$, the positive condition $(n, \ell, m) > 0$ means that either $m > 0$, or $m = 0$ and $n > 0$, or $m = n = 0$ and

Reflective Borcherds products of singular weight

$\ell > 0$, and the Weyl vector $\rho = (A, \vec{B}, C)$ of $\mathbf{B}(\phi)$ is defined by

$$A = \frac{1}{24} \sum_{\ell \in L'} f(0, \ell), \quad \vec{B} = \frac{1}{2} \sum_{\ell > 0} f(0, \ell) \ell, \quad C = \frac{1}{2 \operatorname{rk}(L)} \sum_{\ell \in L'} f(0, \ell) (\ell, \ell).$$

The Fourier–Jacobi expansion of $\mathbf{B}(\phi)$ reads

$$\mathbf{B}(\phi)(Z) = \left(\Theta_{f(0,*)}(\tau, \mathfrak{z}) \cdot \xi^C \right) \exp \left(- \sum_{m=1}^{\infty} \left(\phi|_0 T_-^{(1)}(m) \right) (\tau, \mathfrak{z}) \cdot \xi^m \right),$$

where the leading Fourier–Jacobi coefficient is given by a generalized theta block

$$\Theta_{f(0,*)}(\tau, \mathfrak{z}) = \eta(\tau)^{f(0,0)} \prod_{\ell > 0} \left(\frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)} \right)^{f(0,\ell)}.$$

Hecke operator

Theorem (Clery-Gritsenko)

Let $\varphi \in J_{k,L,t}^1(v_\eta^D)$. Assume $k \in \mathbb{Z}$ and D is an even divisor of 24. If $Q = 24/D$ is odd, we further assume $t \in \mathbb{Z}$. Then for any positive integer m coprime to Q

$$\varphi|_{k,t} T_-^{(Q)}(m)(\tau, \mathfrak{z}) = m^{-1} \sum_{\substack{ad=m, a>0 \\ 0 \leq b < d}} a^k v_\eta^D(\sigma_a) \varphi \left(\frac{a\tau + bQ}{d}, a\mathfrak{z} \right) \in J_{k,L,mt}^1(v_\eta^{Dx}),$$

where $x, y \in \mathbb{Z}$ such that $mx + Qy = 1$, and

$$\sigma_a = \begin{pmatrix} dx + Qdxy & -Qy \\ Qy & a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, one can express Fourier coefficients of $\varphi|_{k,t} T_-^{(Q)}(m)$ in Fourier coefficients of φ as

$$f_m(n, \ell) = \sum_{a|(n, \ell, m)} a^{k-1} v_\eta^D(\sigma_a) f \left(\frac{nm}{a^2}, \frac{\ell}{a} \right).$$

Gritsenko lift

Theorem (Clery-Gritsenko)

Let D be an even divisor of 24, $k \in \mathbb{N}$, $t \in \frac{1}{2}\mathbb{N}$. If $Q = 24/D$ is odd, we further assume that $t \in \mathbb{N}$. Let $\varphi \in J_{k,L,t}(v_\eta^D)$. Let $G_k(\tau)$ be the normalized Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$ whose Fourier coefficient at q is 1. Then the function

$$\mathbf{G}(\varphi)(Z) = f(0,0)G_k(\tau) + \sum_{0 < m \in 1+Q\mathbb{Z}} \left(\varphi|_{k,t} T_-^{(Q)}(m) \right) (\tau, \mathfrak{z}) \cdot \xi^{m/Q}$$

is a modular form of weight k and some character on $2U \oplus L(Qt)$. The form $\mathbf{G}(\varphi)$ is always invariant under the involution $(\omega, \mathfrak{z}, \tau) \mapsto (\tau, \mathfrak{z}, \omega)$.

Affine Kac-Moody algebras and WZW models

The Wess-Zumino-Witten models are 2d RCFTs with additional conserved currents generating an affine Kac-Moody algebra. They are nonlinear sigma models with group manifold G as target, depending on a **positive integer k** called **level**.

The central charge and conformal weights are

$$\text{WZW } (\mathfrak{g})_k : \quad c = \frac{k \dim(\mathfrak{g})}{h_{\mathfrak{g}}^{\vee} + k}, \quad h_{\lambda} = \frac{\langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle}{2(h_{\mathfrak{g}}^{\vee} + k)} \text{ with } a^{\vee} \cdot \lambda = k.$$

The characters of WZW model $(\mathfrak{g})_k$ are just the **characters of affine Lie algebra $\mathfrak{g}^{(1)}$** computed by Kac-Weyl character formula

$$\widehat{\chi}_{\lambda}^{\mathfrak{g}}(\tau) = q^{-\frac{c}{24} + h_{\lambda}} (\dim(\chi_{\lambda}^{\mathfrak{g}}) + \mathcal{O}(q)).$$

Hyperbolization of affine Kac-Moody algebras

Definition

Let \mathfrak{g} be an affine Kac–Moody algebra. A BKM superalgebra \mathfrak{g} without odd real roots is called a **hyperbolization** of \mathfrak{g} if there exists an even positive definite lattice L such that the root lattice of \mathfrak{g} is $U \oplus L$ and the denominator of \mathfrak{g} defines a Borcherds product F of weight $\text{rk}(L)/2$ on $2U \oplus L$ whose leading Fourier–Jacobi coefficient coincides with the denominator $\theta_{\mathfrak{g}}$ of \mathfrak{g} . We also call F the **hyperbolization** of $\theta_{\mathfrak{g}}$.

Hyperbolization of affine Kac-Moody algebras

Theorem (KS-Wang-Williams 23)

Suppose that there is a reflective Borcherds product F of singular weight on $2U \oplus L$ whose Jacobi form input has non-negative q^0 -term. Then (a) or (b) below holds.

- (a) L is the Leech lattice and F is the denominator Φ_{12} of the fake monster algebra.
- (b) There exists a semi-simple Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^s \mathfrak{g}_{j,k_j}$ of rank $\text{rk}(L)$ satisfying restrictions (1) or (1'). Let $\lambda \in U$ with $\lambda^2 = 2$.
 - (1) If F vanishes on λ^\perp , then

$$C := \frac{\dim \mathfrak{g}}{24} - 1 = \frac{h_j^\vee}{k_j}, \quad \text{for } 1 \leq j \leq s. \quad (1)$$

- (1') If F does not vanish on λ^\perp , then

$$C := \frac{\dim \mathfrak{g}}{24} = \frac{h_j^\vee}{k_j} \quad \text{and} \quad k_j > 1, \quad \text{for } 1 \leq j \leq s. \quad (2)$$

Hyperbolization of affine Kac-Moody algebras

Let $c_{\mathfrak{g}}$ denote the central charge of the affine VOA generated by \mathfrak{g} . Then $c_{\mathfrak{g}}$ is always 24 in case (1) and in case (1') we have

$$c_{\mathfrak{g}} = \frac{24C}{C+1}.$$

We call case (1) (resp. (1')) the **anti-symmetric** (resp. **symmetric**) case, since F is anti-invariant (resp. invariant) under the involution $(\omega, \mathfrak{z}, \tau) \mapsto (\tau, \mathfrak{z}, \omega)$.

Necessary conditions for hyperbolization of $\bigoplus \mathfrak{g}_{i,k_i}$

Antisymmetric Constraints for $\bigoplus \mathfrak{g}_{i,k_i}$: **Schellekens' first trace identity!**

$$\frac{1}{24} \sum_i \dim(\mathfrak{g}_i) - 1 = C = \frac{h_i^\vee}{k_i}.$$

The central charge $c = 24$. The weight 0 Jacobi form

$$\phi_{\text{Borch}}|_{m_{\mathfrak{g}} \rightarrow 0} = J(\tau) + N = q^{-1} + N + 196884q + \dots$$

Symmetric Constraints for $\bigoplus (\mathfrak{g}_i)_{k_i}$:

$$\frac{1}{24} \sum_i \dim(\mathfrak{g}_i) = C = \frac{h_i^\vee}{k_i}, \quad k_i > 1.$$

The central charge $c = \frac{24C}{C+1}$. The weight 0 Jacobi form

$$\phi_{\text{Borch}}|_{m_{\mathfrak{g}} \rightarrow 0} = \text{const.}$$

Antisymmetric solutions for $\bigoplus(\mathfrak{g}_i)_{k_i}$, 69 out of 221

C	g	C	g	C	g
1/24	$A_{1,48}^1 A_{2,72}^2$	1/3	$A_{1,6}^2 A_{2,9}^2 B_{2,9}$	1	$A_{1,2} B_{2,3} B_{3,5} G_{2,4}$
1/24	$A_{1,48}^1 B_{2,72}$	1/3	$A_{1,6}^2 B_{2,9}^2$	1	$A_{2,3}^2 A_{4,5}$
1/24	$A_{1,48}^1 A_{2,72} G_{2,96}$	1/3	$A_{1,6}^2 A_{2,9} A_{3,12}$	1	$A_{1,2} A_{4,5} B_{3,5}$
1/24	$A_{3,96} B_{2,72}$	1/3	$A_{2,9}^2$	1	$A_{1,2}^2 A_{2,3} A_{4,5} B_{2,3}$
1/12	$A_{1,24}^1 A_{2,36}$	1/3	$A_{1,6}^2 A_{3,12} G_{2,12}$	1	$A_{2,2}^2 B_{2,5}^2$
1/12	$A_{1,24}^1 G_{2,48}$	1/2	$A_{2,6}^2 G_{2,8}^2$	1	$A_{3,2} B_{2,3} C_{3,4} G_{2,4}$
1/12	$A_{1,24}^1 B_{2,36}^2$	1/2	$A_{1,4}^2 A_{3,8}^2$	1	$A_{1,2}^2 A_{2,3}^2 A_{3,4}$
1/12	$A_{2,36}^2 B_{2,36}$	1/2	$A_{1,4}^2 A_{2,6}^2$	1	$A_{1,2} A_{3,4}$
1/12	$A_{1,24} A_{2,36} A_{3,48}$	1/2	$A_{1,4}^2 A_{4,10}$	1	$A_{1,2}^2 A_{2,3}$
1/8	$A_{1,16} B_{2,24} G_{2,32}$	1/2	$A_{1,4}^2 A_{2,6} A_{3,8} B_{2,6}$	1	$A_{1,2}^2 A_{2,3}$
1/8	$A_{1,16} A_{2,24}^2$	1/2	$A_{1,4}^2 A_{3,8}$	3/2	$A_{2,2}^2 D_{4,4}$
1/8	$A_{1,16}^2 C_{3,32}$	1/2	$C_{4,10}$	3/2	$B_{2,2}^2$
1/8	$A_{1,16}^2 A_{2,24} B_{2,24}$	1/2	$A_{1,4}^2 A_{3,10}$	3/2	$A_{2,2}^2 F_{1,6}$
1/8	$A_{1,16}^2$	1/2	$A_{1,2}^2 B_{2,3} G_{2,4}$	2	$A_{1,3}^2 C_{3,2} D_{8,4}$
1/8	$A_{1,16}^2 A_{3,32}$	1/2	$A_{1,2}^2 A_{2,3}^2 C_{3,4}$	2	$A_{1,3}^2 C_{3,2}^2$
1/8	$A_{1,16}^2 B_{3,40}$	1/2	$A_{1,2}^2 A_{2,3} D_{4,6}$	2	$A_{1,1}^2 A_{3,2} C_{3,2}^2$
1/8	$A_{1,16}^2 A_{4,40}$	1/2	$A_{1,4}^2 A_{2,6}^2 G_{2,8}$	2	$A_{1,1}^2 D_{5,4}$
1/6	$A_{1,12}^1 A_{2,18} G_{2,24}$	1/2	$B_{4,14}$	2	$A_{1,1}^2 G_{2,2}^2$
1/6	$A_{1,12}^1 B_{2,18}$	1/2	$A_{1,4}^2$	2	$A_{1,4}^2 A_{2,2}^2$
1/6	$A_{1,12}^1 A_{2,18}^2$	1/2	$A_{1,4}^2 B_{2,6} G_{2,8}$	2	$A_{1,1}^2 C_{3,2}^2$
1/6	$D_{12,36}^2$	1/2	$A_{2,6}^2 D_{4,12}$	2	$A_{1,1}^2 A_{3,2}$
1/6	$G_{2,24}^2$	1/2	$A_{1,4}^2 C_{3,8}$	2	$A_{1,1}^2 G_{2,2}^2$
1/6	$A_{1,12}^1 A_{3,24} B_{2,18}$	1/2	$A_{1,4}^2 B_{2,6}^2$	2	$A_{1,1}^2 A_{3,2} G_{2,2}^2$
1/6	$A_{2,18} B_{2,18}^2$	1/2	$A_{3,8}^2 C_{3,8}$	2	$A_{1,1}^2 A_{3,2}^2 D_{4,3}$
1/4	$A_{1,8}^1 C_{3,16}$	2/3	$A_{1,3}^2 G_{2,6}^2$	2	$A_{1,1}^2 A_{3,2}^2$
1/4	$A_{1,8}^1$	2/3	$A_{1,3}^2 D_{4,9}$	2	$A_{1,1}^2 A_{3,2}^2$
1/4	$A_{1,8}^1 A_{2,12}^2$	3/4	$A_{2,4}^2 B_{2,4}$	2	$A_{1,1}^2 A_{3,2}^2$
1/4	$A_{1,8}^1 A_{3,16}$	1	$A_{1,2}^2 A_{2,3}^2 B_{2,3}^2$	2	$A_{1,1}^2 C_{3,2} D_{4,3} G_{2,2}$
1/4	$A_{1,8}^1 B_{2,12} G_{2,16}$	1	$A_{2,2}^2 B_{2,3}^2$	2	$A_{1,1}^2 A_{7,4}$
1/4	$A_{1,8}^1 B_{3,20}$	1	$B_{2,2}^2 G_{2,4}^2$	2	$A_{1,1}^2 D_{6,5}$
1/4	$A_{1,8}^1$	1	$A_{1,2}^2 C_{3,4}$	2	$A_{1,1}^2 A_{3,2} D_{4,3} G_{2,2}$
1/4	$A_{1,8}^1 A_{4,20}$	1	$A_{1,2}^2 A_{3,4} B_{3,5}$	2	$A_{1,1}^2 C_{3,2} G_{2,2}^2$
1/4	$A_{1,8}^1 A_{12,12} B_{2,12}$	1	$A_{4,5} B_{2,3} G_{2,4}^2$	2	$A_{1,1}^2 C_{3,2}^2$
1/4	$A_{2,12}^2 G_{2,16}$	1	$A_{1,2}^2 A_{2,3}^2 A_{3,4} G_{2,4}$	2	$A_{1,1}^2 A_{4,3} G_{2,2}$
1/4	$B_{2,12}^2$	1	$A_{1,2}^2 C_{4,5}$	2	$A_{1,1}^2 A_{2,3}$
1/3	$A_{2,9}^2 B_{2,9} G_{2,12}$	1	$A_{1,2}^2$	2	$A_{1,1}^2 C_{3,2}$
1/3	$A_{1,6}^2 A_{2,9}$	1	$A_{1,2}^2 A_{5,6} B_{2,3}$	2	$A_{1,1}^2 C_{3,2} G_{2,2}$
1/3	$A_{1,6}^2 A_{2,9} C_{3,12}$	1	$A_{1,2}^2 D_{5,8}$	2	$A_{1,1}^2 C_{3,2}^2$
1/3	$A_{2,9} A_{4,15}$	1	$A_{1,2}^2 A_{2,3} G_{2,4}$	2	$A_{1,1}^2 A_{3,2} C_{3,2}$
1/3	$A_{1,6}^2 A_{2,9} B_{3,15}$	1	$A_{1,2}^2 A_{2,3} B_{2,3} G_{2,4}$	2	$A_{1,1}^2 A_{3,2} C_{3,2}$
1/3	$A_{1,6}^2 G_{2,12}$	1	$A_{1,2}^2 D_{1,6} G_{2,4}$	2	$A_{1,1}^2 A_{3,2} C_{3,2}$
				4/3	$C_{2,3}^2 G_{2,3}$
				4/3	$G_{2,3}^2$

TABLE 3. The 221 solutions of equation (4.1) in the order of increasing C. Those allowing hyperbolization are colored blue (continued on next page).

C	g	C	g	C	g
3/2	$A_{1,2}^2 B_{2,2}^2$	5/2	$A_{2,2}^2 C_{4,2}$	5	$B_{2,1}^2 C_{4,1} D_{6,2}$
3/2	$A_{2,2}^2 D_{4,4}$	5/2	$B_{2,2}^2$	11/2	$B_{2,2}^2$
3/2	$B_{2,2}^2$	3	$A_{2,1}^2 A_{8,3}$	6	$D_{4,1}^2$
3/2	$A_{2,2}^2 F_{1,6}$	3	$A_{2,1}^2 B_{2,1} E_{6,4}$	6	$A_{3,1} C_{5,1} E_{6,2}$
2	$A_{1,3}^2 C_{3,2} D_{8,4}$	3	$A_{2,1}^2 D_{2,2}^2$	6	$A_{3,1} E_{7,3}$
2	$A_{1,3}^2 C_{3,2}^2$	3	$A_{2,1}^2 B_{2,1}^2$	6	$A_{3,1}^2 D_{4,1}$
2	$A_{1,1}^2 A_{3,2} C_{3,2}^2$	3	$A_{2,1}^2 B_{2,1}^2$	7	$B_{2,1}^2 D_{6,2}$
2	$A_{1,1}^2 D_{5,4}$	3	$A_{2,1}^2 B_{2,1}^2 D_{4,2}$	7	$B_{4,1} C_{6,1}^2$
2	$A_{1,1}^2 G_{2,2}^2$	3	$A_{2,1}^2$	7	$A_{6,1}^2$
2	$A_{1,4}^2 A_{2,2}^2$	3	$A_{2,1}^2 D_{4,2} F_{4,3}$	7	$A_{6,1} B_{4,1}^2$
2	$A_{1,1}^2 C_{3,2}^2$	3	$A_{2,1}^2 B_{2,1}^2 F_{4,3}$	8	$A_{7,1} D_{9,2}$
2	$A_{1,1}^2 A_{3,2}$	3	$B_{2,1}^2 D_{2,2}^2$	8	$A_{2,1}^2 D_{2,1}^2$
2	$A_{1,1}^2 A_{3,2} G_{2,2}^2$	3	$A_{2,1}^2 A_{2,3}^2 B_{2,1}$	9	$C_{8,1} F_{2,1}^2$
2	$A_{1,1}^2 A_{3,2}^2$	3	$A_{2,1}^2 B_{2,1}^2 D_{4,2}$	9	$B_{5,1} E_{7,2} F_{4,1}$
2	$A_{1,1}^2 A_{3,2}^2$	7/2	$B_{2,2}^2$	9	$A_{8,1}^2$
2	$A_{1,1}^2 C_{7,2}$	4	$A_{3,1} C_{7,2}$	10	$D_{6,1}^2$
2	$A_{3,1}^2 D_{5,2}$	4	$A_{3,1}^2 D_{5,2}$	10	$A_{5,1}^2 D_{6,1}$
2	$A_{3,1}^2 C_{10,1}$	4	$A_{3,1}^2 C_{10,1}$	11	$B_{6,1} C_{10,1}$
2	$A_{3,1}^2 D_{5,2}^2$	4	$A_{3,1}^2 A_{7,2} C_{3,1}^2$	23/2	$B_{1,2}^2$
2	$A_{3,1}^2 A_{7,2} C_{3,1}^2$	4	$C_{2,1}^2 E_{6,3}$	12	$E_{6,1}^2$
2	$A_{1,1}^2 A_{3,2} D_{4,3} G_{2,2}$	4	$A_{3,1}^2$	12	$A_{11,1} D_{7,1} E_{6,1}$
2	$A_{1,1}^2 C_{3,2} G_{2,2}^2$	4	$A_{3,1}^2$	13	$A_{1,2}^2$
2	$A_{1,1}^2 C_{3,2}^2$	4	$A_{3,1}^2 C_{3,1} G_{2,1}^2$	14	$D_{8,1}^2$
2	$A_{1,1}^2 C_{3,2}^2$	4	$A_{3,1}^2 A_{7,2} G_{2,1}^2$	15	$B_{8,1} E_{8,2}$
2	$A_{1,1}^2 A_{4,3} G_{2,2}$	4	$A_{3,1}^2 D_{7,2} G_{2,1}$	16	$A_{15,1} D_{9,1}$
2	$A_{1,1}^2 A_{2,3}$	4	$A_{3,1}^2 C_{3,1}^2 G_{2,1}$	18	$D_{10,1} E_{7,1}$
2	$A_{1,1}^2 C_{3,2}$	4	$E_{6,3} C_{3,1}^2$	18	$A_{17,1} E_{7,1}$
2	$A_{1,1}^2 C_{3,2} G_{2,2}$	4	$A_{3,1}^2$	22	$D_{12,1}^2$
2	$A_{1,1}^2 C_{3,2}^2$	5	$A_{4,1} B_{3,1} C_{4,1}$	25	$A_{24,1}$
2	$A_{1,1}^2 A_{3,2} C_{3,2}$	5	$A_{4,1}^2$	30	$E_{8,1}^2$
2	$A_{1,1}^2 A_{3,2} C_{3,2}$	5	$A_{4,1}^2 C_{4,1}$	30	$D_{16,1} E_{8,1}$
2	$A_{1,1}^2 A_{3,2} C_{3,2}$	5	$A_{4,1}^2 A_{9,2} B_{3,1}$	46	$D_{24,1}$
2	$A_{1,1}^2 D_{4,3} G_{2,2}$	5	$C_{4,1}^2$		
2	$A_{1,1}^2 A_{3,2} D_{5,4}$				

TABLE 3. (continued).

Schellenkens' list

In 1993, Schellenkens classified holomorphic CFTs of $c = 24$. There are in total 71 of them:

- 1 Monster CFT, $N = 0$
- 2 Leech CFT, $N = 24$
- 3 69 cases with affine Kac-Moody structures, $N \geq 36$

- $N = 0$ case \rightarrow **Monster Lie algebra**
- $N = 24$ case \rightarrow **fake Monster Lie algebra**
- The 69 cases corresponding to 11 conjugacy classes of Conway group C_{00} , 8 of which have been constructed to associate to BKM algebras by **Höhn, Scheithauer, Gritsenko...**
- For existence part, we only need to deal with the rest 3 conjugacy classes.
- For the non-existence part, we use the forbidden component technique in the theory of Borcherds products.

Remarks for the antisymmetric case

After laborious works in lattice theory and Borcherds products for all 221 solutions, we prove **surprisingly**

The affine Lie algebras allowing antisymmetric hyperbolization are **one to one corresponding** to the 69 affine structures in Schellenkens' list and

$$\phi_{\text{Borch}} = \chi_V = q^{-1} + \sum_{\alpha \in \Delta_{\mathfrak{g}}} e^{2\pi i \langle \alpha, \mathfrak{z} \rangle} + \text{rk}(\mathfrak{g}) + O(q).$$

Theorem (KS-Wang-Williams 23)

Let V be a holomorphic VOA of central charge 24 with semi-simple $V_1 = \mathfrak{g}$. Let $L_{\mathfrak{g}}$ denote Höhn's orbit lattice of \mathfrak{g} and χ_V denote the full character of V . Then the $\mathbf{B}(\chi_V)$ of χ_V is a reflective Borcherds product of singular weight on $2U \oplus L_{\mathfrak{g}}$. Moreover, the leading Fourier–Jacobi coefficient of $\mathbf{B}(\chi_V)$ coincides with the denominator of the affine Kac–Moody algebra \mathfrak{g} .

A simple example: $B_{12,2}$, Schellekens' list No.57

The holomorphic CFT character can be expressed by affine characters as

$$\chi_V = \chi_{0,0}^{B_{12,2}} + \chi_{w_1+w_{12},2}^{B_{12,2}} + \chi_{w_{10},3}^{B_{12,2}} + \chi_{w_5,2}^{B_{12,2}}.$$

Decompose the reps into Weyl orbits with norm defined by $(\cdot, \cdot)_{B_{12}}/2$. Then $\chi_V = q^{-1} + (O_{w_2,1} + O_{w_1, \frac{1}{2}} + 12) + \sum_{i=1}^{\infty} c_i q^i$. We calculate

$$c_1 = O_{2w_2,4} + O_{w_1+w_3,3} + O_{w_5, \frac{5}{2}} + O_{w_1+w_{12}, \frac{5}{2}} + O_{w_1+w_2, \frac{5}{2}} + 4O_{w_4,2} \\ + 12O_{2w_1,2} + 12O_{w_3, \frac{3}{2}} + 12O_{w_{12}, \frac{3}{2}} + 44O_{w_2,1} + 90O_{w_1, \frac{1}{2}} + 300.$$

Clearly all orbits in c_1 with norm > 2 have coefficients 1.

$$c_2 = O_{w_{10},5} + O_{2w_0+w_1,5} + O_{w_2+w_4,5} + O_{w_9, \frac{9}{2}} + O_{3w_1, \frac{9}{2}} + O_{w_2+w_3, \frac{9}{2}} + O_{2w_1+w_{12}, \frac{9}{2}} \\ + O_{w_3+w_{12}, \frac{9}{2}} + O_{w_1+w_6, \frac{9}{2}} + 4O_{w_8,4} + 12O_{2w_2,4} + 4O_{w_1+w_5,4} + 12O_{w_7, \frac{7}{2}} \\ + 12O_{w_2+w_{12}, \frac{7}{2}} + 12O_{w_1+w_4, \frac{7}{2}} + 32O_{w_6,3} + 44O_{w_1+w_3,3} + 90O_{w_5, \frac{5}{2}} \\ + 90O_{w_1+w_{12}, \frac{5}{2}} + 90O_{w_1+w_2, \frac{5}{2}} + 224O_{w_4,2} + 288O_{2w_1,2} \\ + 520O_{w_{12}, \frac{3}{2}} + 520O_{w_3, \frac{3}{2}} + 1242O_{w_2,1} + 2535O_{w_1, \frac{1}{2}} + 5792.$$

All orbits with norm > 4 in c_2 have coefficients 1. This implies **singular weight!**

Symmetric solutions for $\bigoplus(\mathfrak{g}_i)_{k_i}$, 12 out of 17

C	\mathfrak{g}
1/8	$A_{1,16}$
1/4	$A_{1,8}^2$
1/3	$A_{2,9}$
1/2	$A_{1,4}^4$
3/4	$A_{2,4}B_{2,4}$
1	$A_{1,2}^8$

C	\mathfrak{g}
1	$A_{2,3}^3$
1	$A_{4,5}$
1	$A_{3,4}A_{1,2}^3$
1	$B_{2,3}G_{2,4}$
1	$B_{2,3}A_{2,3}A_{1,2}^2$
1	$B_{3,5}A_{1,2}$

C	\mathfrak{g}
1	$C_{3,4}A_{1,2}$
3/2	$A_{2,2}D_{4,4}$
3/2	$A_{2,2}^2B_{2,2}^2$
5/2	$A_{4,2}C_{4,2}$
7/2	$A_{6,2}B_{4,2}$

Main Theorem

The affine Lie algebras allowing symmetric hyperbolization are

$C = 1$ Eight affine structures in F_{24} holomorphic SCFT of $c = 12$

$C < 1$ Four exotic cases related to exceptional modular invariants

2d holomorphic SCFTs with $c = 12$

2d holomorphic SCFTs/self-dual vertex operator superalgebras with $c = 12$ only have three types ([Creutzig-Duncan-Riedler 18](#))

- 1 supersymmetric $E_{8,1}$
- 2 Conway SCFT
- 3 24 free chiral fermions F_{24}

F_{24} allows 8 affine Kac-Moody structures ([Harrison-Paquette-Persson-Volpato 20](#))

$$A_{1,2}^8, A_{2,3}^3, A_{4,5}, A_{3,4}A_{1,2}^3, B_{2,3}G_{2,4}, B_{2,3}A_{2,3}A_{1,2}^2, B_{3,5}A_{1,2}, C_{3,4}A_{1,2}.$$

These 8 affine Kac-Moody algebras can be conformally embedded in $SO(24)_1$.

2d holomorphic SCFTs with $c = 12$

The fermionic characters of the eight F_{24} SCFTs are computed as

$$\chi = \eta^{-12} \theta_i^{r/2} \prod_{\alpha \in \Delta_+} \theta_i(z_\alpha), \quad i = 3, 4, 2 \text{ for NS, } \widetilde{\text{NS}}, \text{R.}$$

The $\widetilde{\text{R}}$ sectors have $\chi = 0$. Then the fermionic partition function is

$$Z_{\text{F}} = \frac{1}{2} (|\chi_{\text{NS}}|^2 + |\chi_{\widetilde{\text{NS}}}|^2 + |\chi_{\text{R}}|^2).$$

The input Jacobi form of Borcherds product is given by

$$\phi_{\mathfrak{g}} = \chi_{\text{NS}} - \chi_{\widetilde{\text{NS}}} - \chi_{\text{R}}.$$

We have the other universal formula by theta block

$$\phi_{\mathfrak{g}} = -\frac{\theta_{\mathfrak{g}}|_{\text{rk}(\mathfrak{g})/2} T_-^{(1)}(2)(\tau, \mathfrak{z})}{\theta_{\mathfrak{g}}(\tau, \mathfrak{z})}.$$

Exceptional modular invariants for F_{24}

Consider conformal embedding $\iota : \mathfrak{g} \subset SO(24)_1$, then

$$\chi_{\text{NS}}(\tau, \mathfrak{z}) = \chi_0(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) + \chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})),$$

$$\chi_{\widetilde{\text{NS}}}(\tau, \mathfrak{z}) = \chi_0(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) - \chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})),$$

$$\chi_{\text{R}}(\tau, \mathfrak{z}) = \chi_s(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) + \chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})).$$

The diagonal modular invariant of F_{24} with \mathfrak{g} is given by

$$\begin{aligned} Z_{F_{24}} &= \frac{1}{2} \left(|\chi_{\text{NS}}|^2 + |\chi_{\widetilde{\text{NS}}}|^2 + |\chi_{\text{R}}|^2 + |\chi_{\widetilde{\text{R}}}|^2 \right) \\ &= |\chi_0(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + |\chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + |\chi_s(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + |\chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2. \end{aligned}$$

Recall the triality among v, s, c , by acting the **automorphism** $\chi_v \leftrightarrow \chi_c$ on the **holomorphic part**, we obtain an exceptional modular invariant defined as

$$Z_{F_{24}}^{\text{exc}} = |\chi_0(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + |\chi_s(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + (\chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))\overline{\chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))}) + c.c.).$$

The following relation is immediate

$$Z_{F_{24}} - Z_{F_{24}}^{\text{exc}} = |\phi_{\mathfrak{g}}|^2.$$

Example: $A_{1,2}^8$

The BKM algebra and the reflective Borcherds product associated to $A_{1,2}^8$ were studied by (Borcherds 90s). The relation to $c = 12$ holomorphic SCFT was studied in (Harrison-Paquette-Persson-Volpato 20). The **fermionization** for $A_{1,2}$ is

$$\chi_{\text{NS}}^{A_{1,2}} = \chi_0 + \chi_{1/2}, \quad \chi_{\widetilde{\text{NS}}}^{A_{1,2}} = \chi_0 - \chi_{1/2}, \quad \chi_{\text{R}}^{A_{1,2}} = \sqrt{2}\chi_{\frac{3}{16}}.$$

For $A_{1,2}^8$, naturally

$$\chi_{\text{NS}} = \prod_8 \chi_{\text{NS}}^{A_{1,2}}, \quad \chi_{\widetilde{\text{NS}}} = \prod_8 \chi_{\widetilde{\text{NS}}}^{A_{1,2}}, \quad \chi_{\text{R}} = \prod_8 \chi_{\text{R}}^{A_{1,2}}.$$

We find the following formula for the Jacobi form of the Borcherds product

$$\phi_{A_{1,2}^8} = \frac{1}{2}(\chi_{\text{NS}} - \chi_{\widetilde{\text{NS}}} - \chi_{\text{R}}) = \sum_8 \chi_0^7 \chi_{\frac{1}{2}} + \sum_{56} \chi_0^5 \chi_{\frac{3}{2}}^3 + \sum_{56} \chi_0^3 \chi_{\frac{5}{2}}^5 + \sum_8 \chi_0 \chi_{\frac{7}{2}}^7 - 8\chi_{\frac{3}{16}}^8.$$

The summations are over permutations. It is easy to check $\phi_{A_{1,2}^8}|_{3 \rightarrow 0} = 24$.

Example: $A_{1,2}^8$

Interestingly, $A_{1,2}^8$ also appeared in (Gannon 94) in the study of exceptional modular invariant from non-trivial automorphism of fusion algebras.

$$\begin{aligned} Z_F &= \frac{1}{2} (|\chi_{NS}|^2 + |\chi_{\widetilde{NS}}|^2 + |\chi_R|^2) \\ &= \frac{1}{2} (|\sum_{\text{even}} + \sum_{\text{odd}}|^2 + |\sum_{\text{even}} - \sum_{\text{odd}}|^2 + |16 \prod_8 \chi_{\frac{3}{16}}|^2) \\ &= |\sum_{\text{even}}|^2 + |\sum_{\text{odd}}|^2 + 128 |\prod_8 \chi_{\frac{3}{16}}|^2 \\ &= |\sum_{\text{even}}|^2 + |\sum_{\text{odd}}|^2 + 2|8 \prod_8 \chi_{\frac{3}{16}}|^2 \\ &= |\phi_0|^2 + |\phi_v|^2 + |\phi_s|^2 + |\phi_c|^2. \end{aligned}$$

Exchange $\omega : \phi_v \leftrightarrow \phi_c$ gives an automorphism. Consider

$$Z_{exc} = Z_F \Big|_{\omega \text{ in holomorphic sector}}$$

With $\phi_{\text{Borch}} = \phi_v - \phi_c$, one has $Z_F - Z_{exc} = |\phi_{\text{Borch}}|^2$

Example: $A_{2,3}^3$

The BKM algebra and the reflective Borcherds product associated to $A_{2,3}^3$ were studied by (Gritsenko 10). We utilize the well-known **conformal embedding** $A_{2,3} \subset D_{4,1}$. Let us denote

$$\phi_0 = \chi_{00,0}^{A_{2,3}} + \chi_{03,1}^{A_{2,3}} + \chi_{30,1}^{A_{2,3}}, \quad \phi_1 = \chi_{11, \frac{1}{2}}^{A_{2,3}}.$$

As a SVOA, the fermionization of $A_{2,3}$ has fermionic characters as

$$\chi_{\text{NS}}^{A_{2,3}} = \phi_0 + \phi_1, \quad \chi_{\widetilde{\text{NS}}}^{A_{2,3}} = \phi_0 - \phi_1, \quad \chi_{\text{R}}^{A_{2,3}} = 2\phi_1.$$

Therefore, for holomorphic SVOA $A_{2,3}^3$, we have the fermionic characters

$$\chi_{\text{NS}} = \bigotimes \chi_{\text{NS}}^{A_{2,3}}, \quad \chi_{\widetilde{\text{NS}}} = \bigotimes \chi_{\widetilde{\text{NS}}}^{A_{2,3}}, \quad \chi_{\text{R}} = \bigotimes \chi_{\text{R}}^{A_{2,3}}.$$

where the tensor products take over all three copies of $A_{2,3}$.

Example: $A_{4,5}$

The BKM algebra and the reflective Borcherds product associated to $A_{4,5}$ were studied by (Scheithauer 06). We find the following formula for the input of Borcherds product

$$\phi_{A_{4,5}} = \left(\sum_{\mathbb{Z}_5} \chi_{1001, \frac{1}{2}}^{A_{4,5}} \right) - \chi_{1111, \frac{3}{2}}^{A_{4,5}}.$$

The q -series gives constant 24 which appeared in (Schellekens-Yankielowicz 90) in the search for affine A_4 modular invariant. Let us denote

$$\begin{aligned} \phi_0 &= \sum_{\mathbb{Z}_5} \chi_{0000,0}^{A_{4,5}}, & \phi_1 &= \sum_{\mathbb{Z}_5} \chi_{0102,1}^{A_{4,5}}, & \phi_2 &= \sum_{\mathbb{Z}_5} \chi_{1001, \frac{1}{2}}^{A_{4,5}}, \\ \phi_3 &= \sum_{\mathbb{Z}_5} \chi_{0021, \frac{6}{5}}^{A_{4,5}}, & \phi_4 &= \sum_{\mathbb{Z}_5} \chi_{0110, \frac{4}{5}}^{A_{4,5}}, & \phi_5 &= \chi_{1111, \frac{3}{2}}^{A_{4,5}}. \end{aligned}$$

Example: $A_{4,5}$

The simple-current extended invariant in (Schellekens-Yankielowicz 90) is

$$Z_{(B.2)} = \sum_{i=0}^4 |\phi_i|^2 + 5|\phi_5|^2.$$

The exceptional modular invariant in (Schellekens-Yankielowicz 90) is

$$Z_{(B.4)} = |\phi_0|^2 + |\phi_1|^2 + |\phi_3|^2 + |\phi_4|^2 + 4|\phi_5|^2 + (\phi_2\bar{\phi}_5 + c.c.).$$

It is easy to observe their relation

$$Z_{(B.2)} - Z_{(B.4)} = |\phi_{A_{4,5}}|^2.$$

The 4 exotic CFTs

In math literature, there are four more known reflective Borcherds products of singular weights. In affine Lie algebra language, they are

\mathfrak{g}	$A_{1,16}$	$A_{1,8}^2$	$A_{1,4}^4$	$A_{2,9}$
C	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$
c	$\frac{8}{3}$	$\frac{24}{5}$	8	6
rk	1	2	4	2
dim	3	6	12	8
ref	Gritsenko-Nikulin 98	Grit 19	Grit 18	Gritsenko-Skoruppa-Zagier 19

Question

Is there any physical meaning for these four Borcherds products?

The 4 exotic CFTs

Answer

Yes! They are related to some very peculiar **exceptional modular invariants** that come from the **nontrivial automorphism of the fusion algebra** of the simple current extension.

Such peculiarity of $A_{1,16}$ and $A_{2,9}$ was first noticed by (Moore-Seiberg 88)! later for $A_{1,8}^2$ by (Verstegen 90) and for $A_{1,4}^4$ by (Gannon 94).

The nontrivial automorphism of the fusion algebra happens rarely. It can be proved for $A_{1,k}$ this **only** happens at $k = 16$, while for $A_{2,k}$ **only** at $k = 9$.

The input Jacobi form can be expressed universally as

$$\phi_{\mathfrak{g}}(\tau, \mathfrak{z}) = -\frac{(\vartheta_{\mathfrak{g}}|_{\frac{1}{2}\text{rk}(\mathfrak{g})} T_{-}^{(\frac{1}{C})}(1/C + 1))(\tau, \mathfrak{z})}{\vartheta_{\mathfrak{g}}(\tau, \mathfrak{z})}.$$

Example: $A_{1,16}$

Affine A_1 has an ADE classification of modular invariants. The D_{10} modular invariant of $A_{1,16}$, – a simple current extended modular invariant:

$$\begin{aligned} Z_{D_{10}} &= |\chi_0 + \chi_{16}|^2 + |\chi_2 + \chi_{14}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + 2|\chi_8|^2 \\ &= |\phi_0|^2 + |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 + |\phi_4'|^2. \end{aligned}$$

The S -matrix for the six extended fields $\phi_{0,1,2,3,4,4'}$ is

$$\frac{1}{3} \begin{pmatrix} 2 \sin\left(\frac{\pi}{18}\right) & 1 & 2 \cos\left(\frac{2\pi}{9}\right) & 2 \cos\left(\frac{\pi}{9}\right) & 1 & 1 \\ 1 & 2 & 1 & -1 & -1 & -1 \\ 2 \cos\left(\frac{2\pi}{9}\right) & 1 & -2 \cos\left(\frac{\pi}{9}\right) & -2 \sin\left(\frac{\pi}{18}\right) & 1 & 1 \\ 2 \cos\left(\frac{\pi}{9}\right) & -1 & -2 \sin\left(\frac{\pi}{18}\right) & 2 \cos\left(\frac{2\pi}{9}\right) & -1 & -1 \\ 1 & -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & -1 & 2 \end{pmatrix}.$$

(Moore-Seiberg 88) observed a nontrivial automorphism $\omega : \phi_1 \leftrightarrow \phi_4$ for the fusion algebra! This only happens at level $k = 16$!

E_7 modular invariant

From such automorphism, (Moore-Seiberg 88) obtained the E_7 modular invariant:

$$\begin{aligned} Z_{E_7} &= Z_{D_{10}} \Big|_{\omega \text{ in holomorphic sector}} \\ &= |\phi_0|^2 + \phi_1 \bar{\phi}_4 + |\phi_2|^2 + |\phi_3|^2 + |\phi'_4|^2 + \phi_4 \bar{\phi}_1 \\ &= |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + |\chi_8|^2 \\ &\quad + ((\chi_2 + \chi_{14})\bar{\chi}_8 + c.c.) \\ &= Z_{D_{10}} - |\phi_{A_1}|^2. \end{aligned}$$

The weight 0 Jacobi form

$$\phi_{A_1} = \frac{\theta_1(3z)}{\theta_1(z)} = \chi_{2, \frac{1}{9}}^{A_{1,16}} + \chi_{14, \frac{28}{9}}^{A_{1,16}} - \chi_{8, \frac{10}{9}}^{A_{1,16}}$$

is exactly the Jacobi form for Gritsenko-Nikulin's $\Delta_{1/2}(Z)$.

Example: $A_{2,9}$

Affine A_2 has a generalized ADE classification of modular invariants (Gannon 92,94). The \mathcal{D}_9 modular invariant of $A_{2,9}$, – a simple current extended modular invariant:

$$\begin{aligned} Z_{\mathcal{D}_9} &= \left| \sum \chi_{00,0} \right|^2 + \left| \sum \chi_{11, \frac{1}{4}} \right|^2 + \left| \sum \chi_{03, \frac{1}{2}} \right|^2 + \left| \sum \chi_{30, \frac{1}{2}} \right|^2 \\ &\quad + \left| \sum \chi_{22, \frac{2}{3}} \right|^2 + \left| \sum \chi_{14,1} \right|^2 + 3 \left| \chi_{33, \frac{5}{4}} \right|^2 \\ &= |\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 + |\Phi_4|^2 + |\Phi_5|^2 + |\Phi_6|^2 + |\Phi_6'|^2 + |\Phi_6''|^2. \end{aligned}$$

The extended primaries have conformal weights

$$h_i = 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, 1, \left(\frac{5}{4}\right)_3.$$

Example: $A_{2,9}$

The S -matrix is

$$\frac{1}{4} \begin{pmatrix} \frac{1}{3}(2\sqrt{3}-3) & 1 & 1 & 1 & \frac{4}{\sqrt{3}} & \frac{1}{3}(2\sqrt{3}+3) & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1+2i & -1-2i & 0 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1-2i & -1+2i & 0 & -1 & 1 & 1 & 1 \\ \frac{4}{\sqrt{3}} & 0 & 0 & 0 & -\frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{3}(2\sqrt{3}+3) & -1 & -1 & -1 & \frac{4}{\sqrt{3}} & \frac{1}{3}(2\sqrt{3}-3) & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 & 3 & -1 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 & -1 & 3 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 & -1 & -1 & 3 \end{pmatrix}.$$

(Moore-Seiberg 88) observed the **nontrivial automorphism** $\omega : \Phi_1 \leftrightarrow \Phi_6''$ and find the **\mathcal{E}'_9 modular invariant** given by

$$Z_{\mathcal{E}'_9} = Z_{\mathcal{D}_9} \Big|_{\omega \text{ in holomorphic sector}}.$$

This is the last modular invariant of affine A_2 that was found.

Example: $A_{2,9}$

It is easy to see

$$Z_{\mathcal{D}_9} - Z_{\mathcal{E}'_9} = |\phi_{A_{2,9}}|^2,$$

where

$$\phi_{A_{2,9}}(\tau, z) = \chi_{11, \frac{1}{4}}^{A_{2,9}}(\tau, z) + \chi_{17, \frac{9}{4}}^{A_{2,9}}(\tau, z) + \chi_{71, \frac{9}{4}}^{A_{2,9}}(\tau, z) - \chi_{33, \frac{5}{4}}^{A_{2,9}}(\tau, z).$$

When $z \rightarrow 0$, this reduces to the **Macdonald identity** for A_2 :

$$8 = \chi_{11, \frac{1}{4}}^{A_{2,9}}(\tau) + \chi_{17, \frac{9}{4}}^{A_{2,9}}(\tau) + \chi_{71, \frac{9}{4}}^{A_{2,9}}(\tau) - \chi_{33, \frac{5}{4}}^{A_{2,9}}(\tau).$$

Another form of $\phi_{A_{2,9}}$ is

$$\phi_{A_{2,9}} = -\frac{T_3^{(4)} \Theta_1(z_1, z_2)}{\Theta_1(z_1, z_2)},$$

where

$$\Theta_1(z_1, z_2) = \frac{\theta_1(z_1)\theta_1(z_1 - z_2)\theta_1(z_2)}{\eta},$$

$A_{1,8}^2$ and $A_{1,4}^4$

For $A_{1,8}^2$ and $A_{1,4}^4$ we obtain similar relation between the automorphism modular invariants and \mathfrak{g} . We find the weight-0 Jacobi forms

$$\begin{aligned}\phi_{A_{1,8}^2} &= \left(\chi_{0,0}^{A_{1,8}} + \chi_{8,2}^{A_{1,8}}\right) \otimes \left(\chi_{2,\frac{1}{5}}^{A_{1,8}} + \chi_{6,\frac{6}{5}}^{A_{1,8}}\right) + \left(\chi_{2,\frac{1}{5}}^{A_{1,8}} + \chi_{6,\frac{6}{5}}^{A_{1,8}}\right) \otimes \left(\chi_{0,0}^{A_{1,8}} + \chi_{8,2}^{A_{1,8}}\right) \\ &\quad - 2\chi_{4,\frac{3}{5}}^{A_{1,8}} \otimes \chi_{4,\frac{3}{5}}^{A_{1,8}},\end{aligned}$$

$$\begin{aligned}\phi_{A_{1,4}^4} &= \sum_{\text{cyclic}} \left(\chi_{0,0}^{A_{1,4}} + \chi_{4,1}^{A_{1,4}}\right) \otimes \left(\chi_{0,0}^{A_{1,4}} + \chi_{4,1}^{A_{1,4}}\right) \otimes \left(\chi_{0,0}^{A_{1,4}} + \chi_{4,1}^{A_{1,4}}\right) \otimes \chi_{2,\frac{1}{3}}^{A_{1,4}} \\ &\quad - 4 \bigotimes \chi_{2,\frac{1}{3}}^{A_{1,4}},\end{aligned}$$

where the sum contains 4 terms by permutation.

Further generalize to Lie superalgebras

(Gritsenko-Nikulín 98) also observed a quintuple product formula

$$\begin{aligned}\vartheta_{3/2}(\tau, z) &:= \eta(\tau) \frac{\theta_1(\tau, 2z)}{\theta_1(\tau, z)} = \sum_{n \in \mathbb{Z}} \left(\frac{12}{n} \right) q^{\frac{n^2}{24}} \zeta^{\frac{n}{2}} \\ &= q^{\frac{1}{24}} (\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}) \prod_{n=1}^{\infty} (1 + q^n \zeta)(1 + q^n \zeta^{-1})(1 - q^{2n-1} \zeta^2)(1 - q^{2n-1} \zeta^{-2})(1 - q^n)\end{aligned}$$

can define a holomorphic Jacobi form of weight $1/2$ and index $3/2$ for A_1 with character v_η . They constructed an exotic modular form as the additive lift

$$D_{1/2}(Z) := \mathbf{G}(\vartheta_{3/2})(\omega, z, \tau) = \sum_{m=1}^{\infty} \left(\frac{12}{m} \right) \vartheta_{3/2}(\tau, mz) \cdot e^{2\pi i m^2 \omega / 24},$$

This is a modular form of singular weight on $2U \oplus A_1(36)$ which can be realized as a Siegel paramodular form of genus 2 and level 36. They also proved this is a Borcherds product $\mathbf{B}(\phi_{0,36})$ by input

$$\phi_{0,36}(\tau, z) := \frac{\vartheta_{3/2}(\tau, 5z)}{\vartheta_{3/2}(\tau, z)} = \frac{\theta_1(\tau, 10z)\theta_1(\tau, z)}{\theta_1(\tau, 5z)\theta_1(\tau, 2z)} = (\zeta^2 + \zeta^{-2}) - (\zeta + \zeta^{-1}) + 1 + O(q).$$

Further generalize to Lie superalgebras

This exotic Borcherd product is the denominator of an exotic BKM algebra with **odd real roots**. We find it can be regarded as the hyperbolization of **affine Lie superalgebra** $A_{1,36}^*$. Here we use A_1^* to denote $osp(1|2)$. Indeed, we weight-0 Jacobi form of $D_{1/2}(Z)$ can be written as the linear combination of the Kac-Wakimoto characters of $A_{1,36}^*$.

A_1^* has $h^* = 3/2$. The associated theta block is defined as $\vartheta_{A_1^*} = \vartheta_{3/2}$. The associated root lattice is defined as $\mathbf{Q}_{A_1^*} = A_1$. The dimension is defined as 1.

Question ([Gritsenko-Nikulin 98](#))

Is this the unique example of reflective Borchers product of singular weight with odd real roots?

Hyperbolization of affine Lie superalgebras

Answer

No! Recently we are able to construct two new examples.

Theorem (KS-Wang-Williams 24)

If $2U \oplus L$ has a symmetric reflective Borcherds product F of singular weight for which the q^0 -term of the Jacobi form input involves negative Fourier coefficients, L is isomorphic to $A_1(36)$, $A_1(9) \oplus A_1(3)$ or $2A_1(10)$. They are denominator of the hyperbolization of affine Lie superalgebras $A_{1,36}^$, $A_{1,9}^*A_{1,12}$ and $C_{2,10}^*$.*

Here we use C_n^* , $n \geq 2$ to denote Lie superalgebra $osp(1|2n)$.

Thank you!