Seiberg-Witten curves from 5-brane webs with orientifold planes

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This talk is mainly based on
JHEP 11 (2023) 178: Hirotaka Hayashi, Sung-Soo Kim, Kimyeong Lee, F.Y.
JHEP 06 (2021) 004: Xiaobin Li, F.Y.
JHEP 1711 (2017) 041: Hirotaka Hayashi, Sung-Soo Kim, Kimyeong Lee, F.Y.

String Theory and Quantum Field Theory 2024, January 30

Review of
Seiberg-Witten curves

Seiberg-Witten Theory

['94 Seiberg, Witten]

Seiberg-Witten 1-form
on
Seiberg-Witten curve

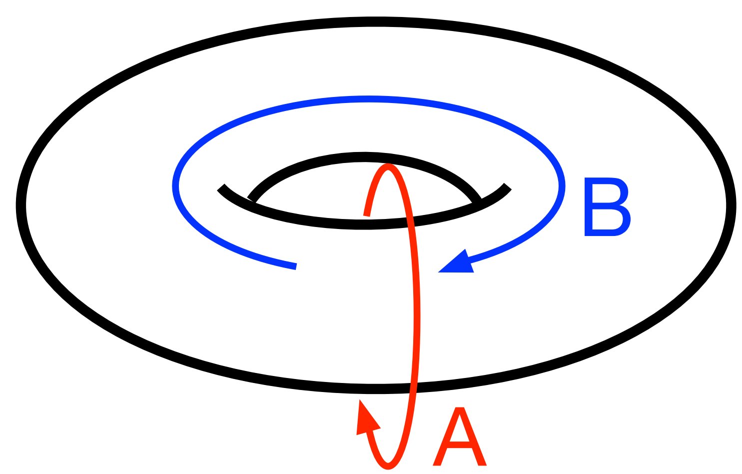
→

Effective Prepotential of
4d N=2 gauge theory
at Coulomb phase

For SU(2) gauge theory

$$a = \oint_A \lambda_{\mathrm{SW}}$$

$$\frac{\partial F(a)}{\partial a} = \oint_B \lambda_{\mathrm{SW}}$$



$F(a)$: Effective prepotential at Coulomb phase

λ_{SW} : Seiberg-Witten 1-form

Various approaches for Seiberg-Witten curve

- **Original field theoretic approach**

  ['94 Seiberg, Witten], …

- **Spectral curve of the integrable system**

  ['95 Gorsky, Krichever, Marshakov, Mironov, Morozov], ['95 Martinec, Warner], ….

- **Geometric engineering (Calabi-Yau compactification)**

  ['96 Katz, Klemm, Vafa], …

- **Brane setup (Hanany-Witten type, 5-brane web)**

  ['97 Witten], …

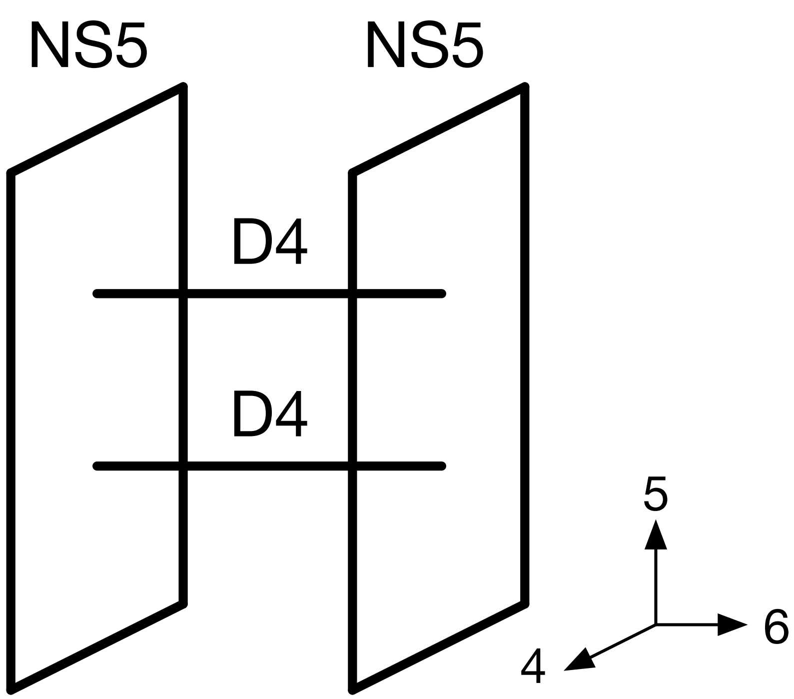
- **6d N=(2,0) on Riemman surface (class S)**

  ['09 Gaiotto], ….

⋮

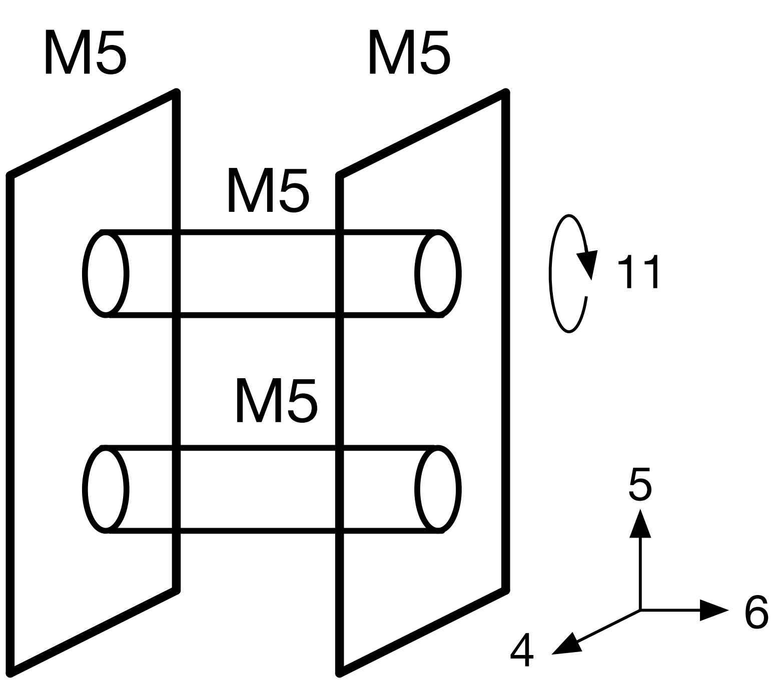
Seiberg-Witten Curve from brane setup ['97 Witten]

Hanany-Witten type brane setup for 4d N=2 SU(N) gauge theory



	0	1	2	3	4	5	6	7	8	9
NS5-brane	—	—	—	—	—	—	•	•	•	•
D4-brane	—	—	—	—	•	•	—	•	•	•

M-theory uplift



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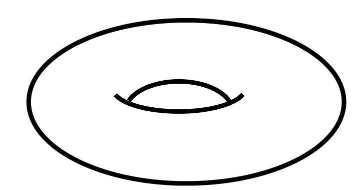
SW curve

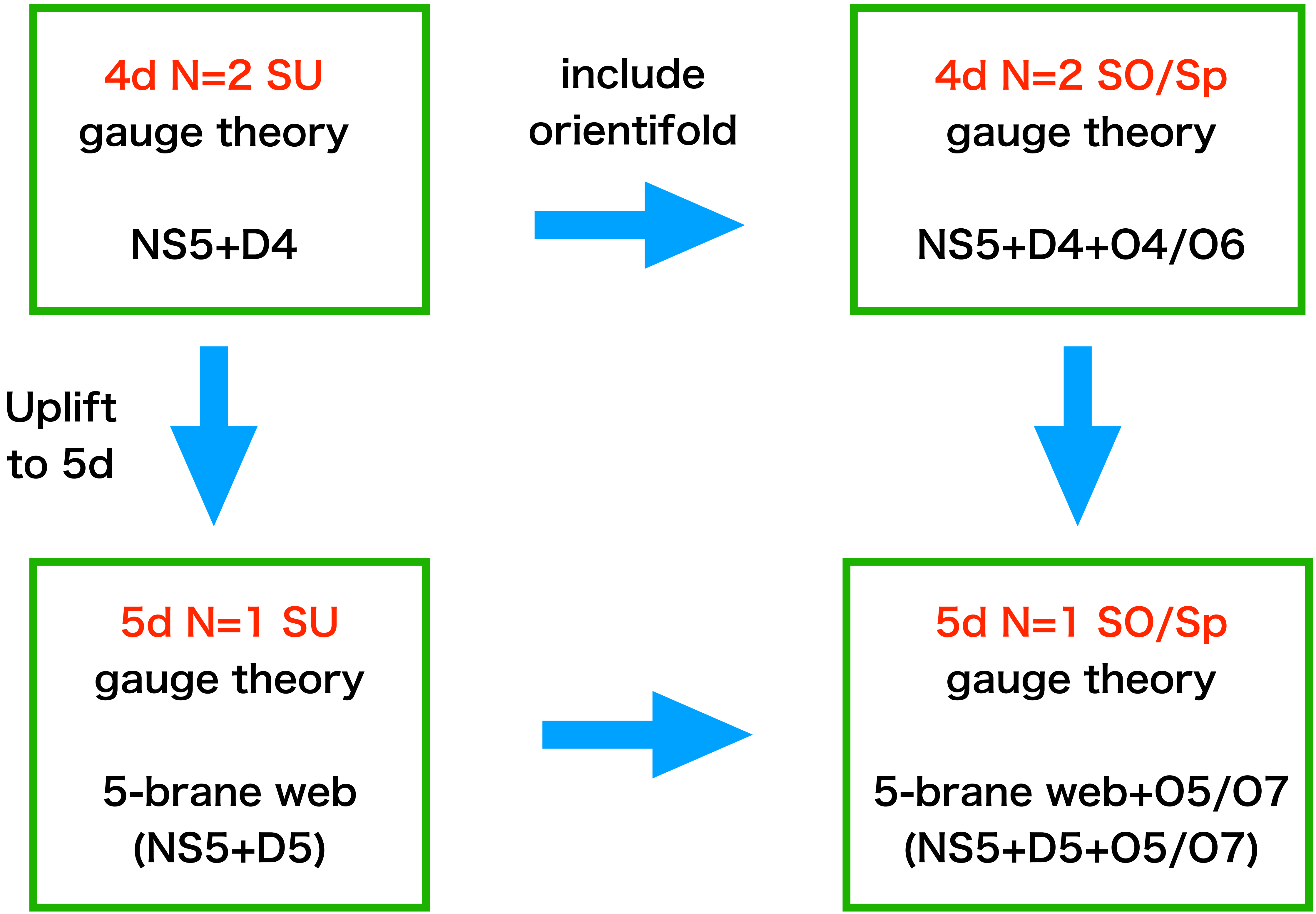


$$t^2 + \underline{p_2(v)}t + 1 = 0$$

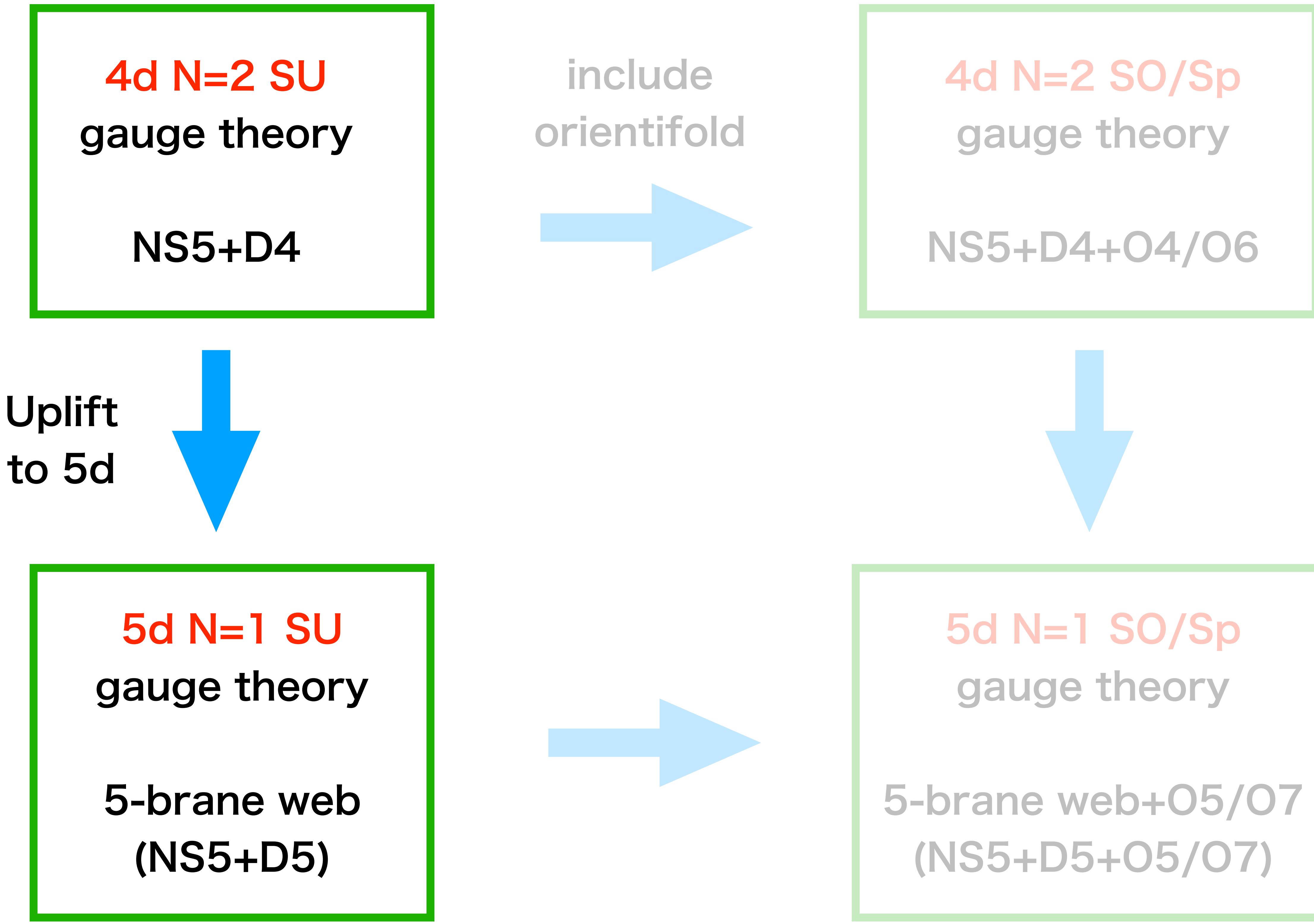
Degree 2 polynomial of v

$$v = x_5 + i x_4, \qquad t = e^{-\frac{x_6 + i x_{11}}{R_M}}$$

Generalization



Generalization



Seiberg-Witten Curve for 5d N=1 gauge theory

['96 Nekrasov]
cf ['96 Seiberg]

5d N=1 gauge theory on S^1

Coulomb phase
(VEV for vector multiplet)

RG flow

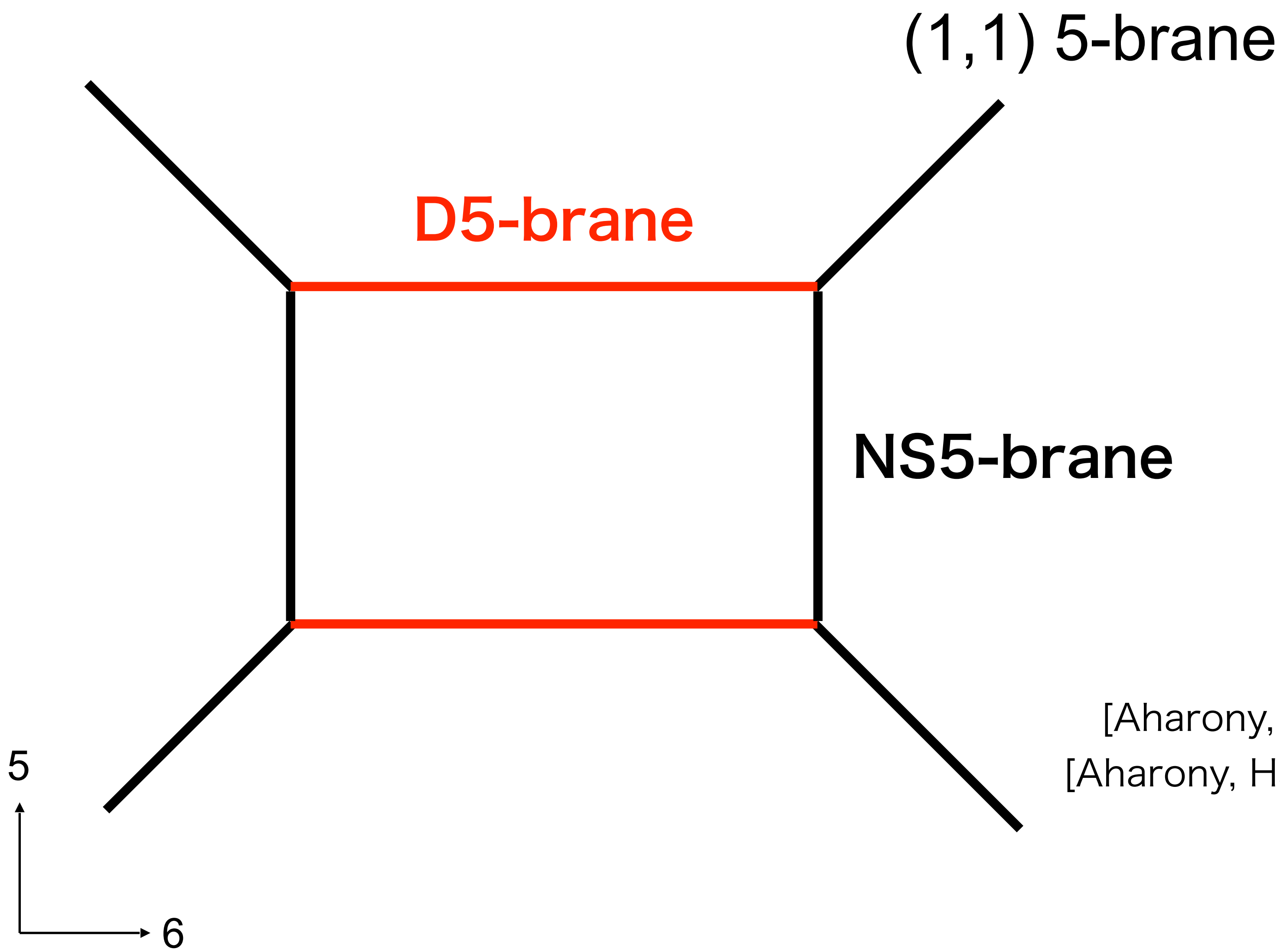
All the Kaluza-Klein modes
are integrated out

4d N=2 Abelian gauge theory

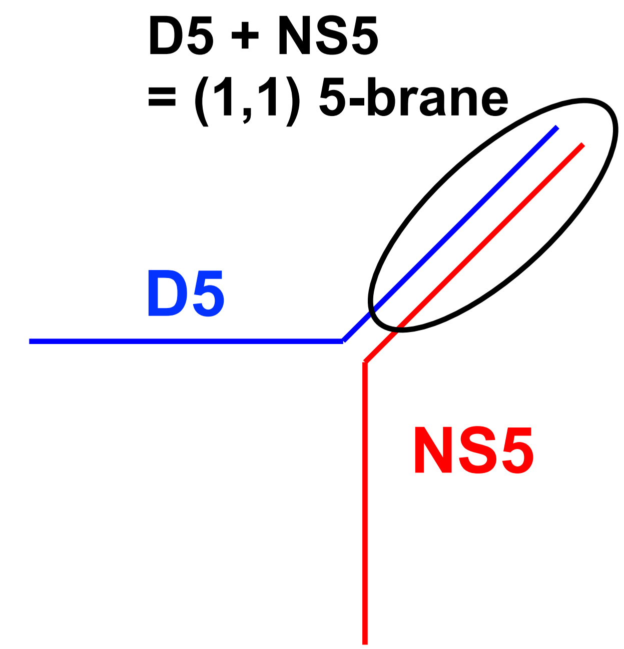
Also in this case, the effective prepotential is given
in terms of **Seiberg-Witten curve**

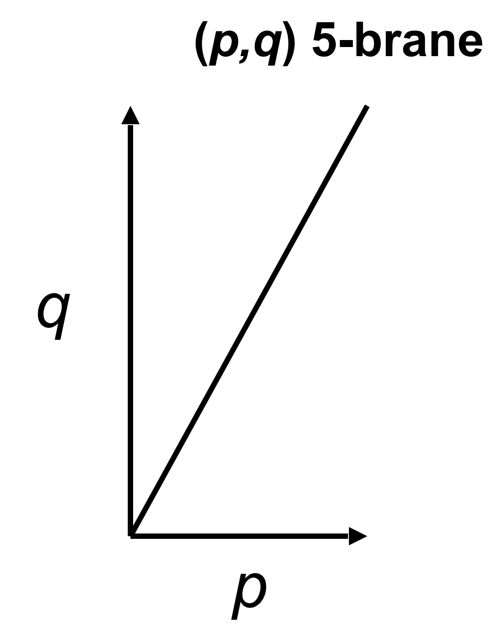
5d N=1 gauge theory from (p,q) 5-brane web

5D N=1 SU(2) SYM



[Aharony, Hanany '97]
[Aharony, Hanany, Kol '97]





	0	1	2	3	4	5	6	7	8	9
NS5-brane	—	—	—	—	—	—	•	•	•	•
D5-brane	—	—	—	—	—	•	—	•	•	•
(p,q) 5-brane	—	—	—	—	—	(p,q)		•	•	•

Seiberg-Witten Curve from 5-brane web

['97 Brandhuber, Itzhaki, Sonnenschein, Theisen, Yankielowicz]

Type IIB: 5-brane web

	0	1	2	3	4	5	6	7	8	9
NS5-brane	—	—	—	—	—	—	•	•	•	•
D5-brane	—	—	—	—	—	•	—	•	•	•
(p,q) 5-brane	—	—	—	—	—	(p,q)		•	•	•

T-duality along direction 4

Type IIA: NS5-D4

	0	1	2	3	4	5	6	7	8	9
NS5-brane	—	—	—	—	—	—	•	•	•	•
D4-brane	—	—	—	—	•	•	—	•	•	•

Uplift

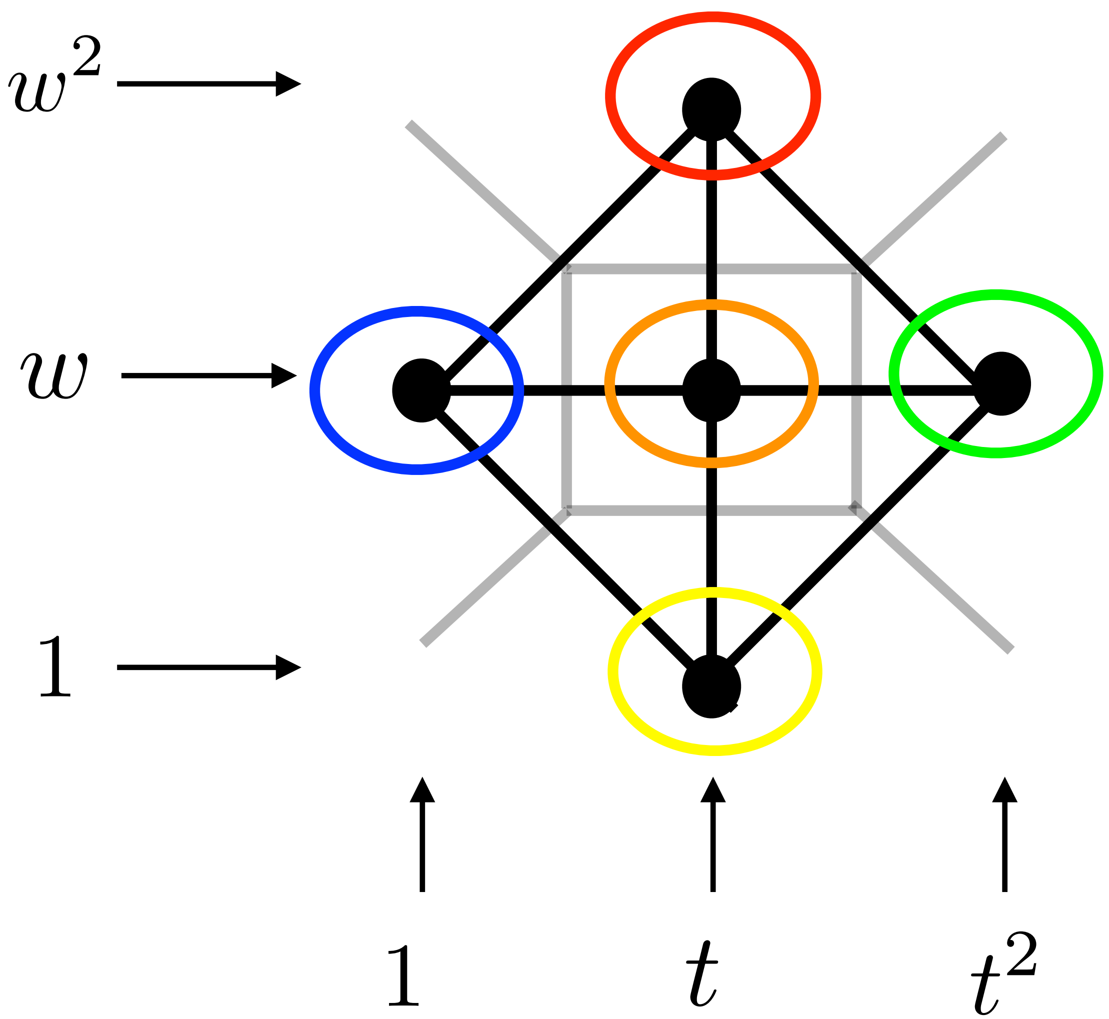
M-theory: M5-brane

SW curve for 5d N=1 gauge theory

Seiberg-Witten Curve from 5-brane web

['97 Aharony, Hanany, Kol]

"Dual graph" of the 5-brane web



Seiberg-Witten curve

$$C_{21}w^2t + C_{10}w + C_{11}wt + C_{12}wt^2 + C_{01}t = 0$$

$$w = e^{-\frac{x_5 + ix_4}{R_A}}, \qquad t = e^{-\frac{x_6 + ix_{11}}{R_M}}$$

SW 1-form: $\lambda_{\mathrm{SW}} \sim \log t \, d \log w$

Various approaches for Seiberg-Witten curve

- **Original field theoretic approach**

  ['94 Seiberg, Witten], …

- **Spectral curve of the integrable system**

  ['95 Gorsky, Krichever, Marshakov, Mironov, Morozov], ['95 Martinec, Warner], ….

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⋮

Original discussion for geometric engineering approach

['96 Katz, Klemm, Vafa],

4d N=2 gauge theory is realized by IIA on (local) CY_3

↓ **Local mirror symmetry**

IIB on (local) CY_3

↓ **"Field theory limit"(?)**

Seiberg Witten curve (geometry)
for 4d N=2 gauge theory

Current understanding for geometric engineering approach

5d N=1 gauge theory on S^1 is realized
by IIA on (local) CY_3 = M on S^1 x (local) CY_3

↓ **Local mirror symmetry**

IIB on (local) CY_3
= Seiberg Witten curve (geometry)
for 5d N=1 gauge theory on S^1

cf ['00 Eguchi, Kanno]

↓ ~~"Field theory limit"(?)~~
"4d limit"

Seiberg Witten curve (geometry)
for 4d N=2 gauge theory

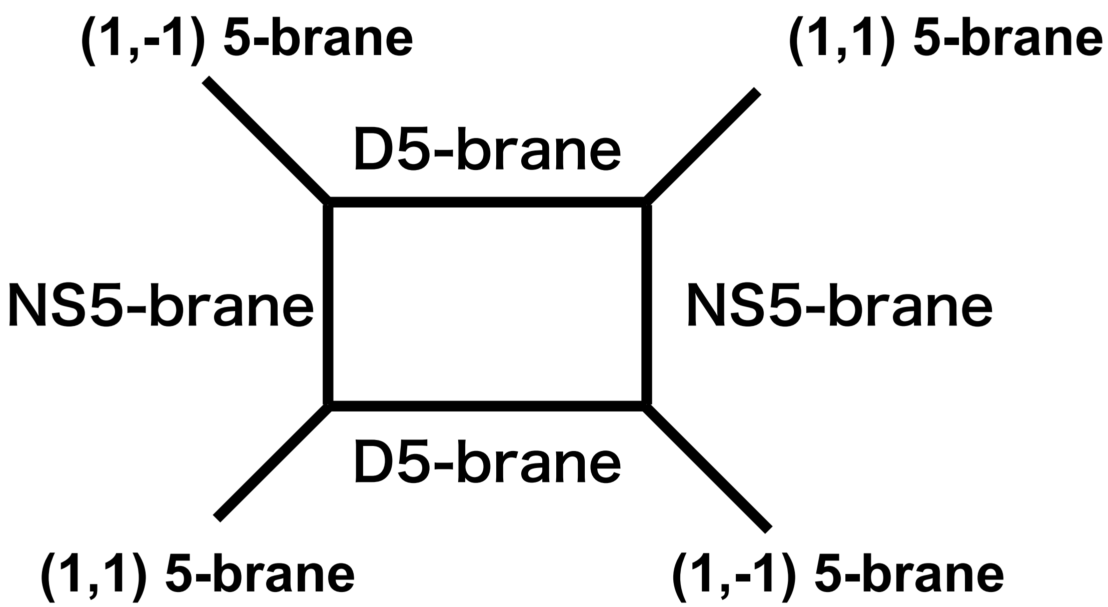
(Local) Calabi-Yau 3-fold "describes" 5d N=1 gauge theory on S^1

Local mirror dual gives SW curve (geometry) for 5d N=1 gauge theory on S^1

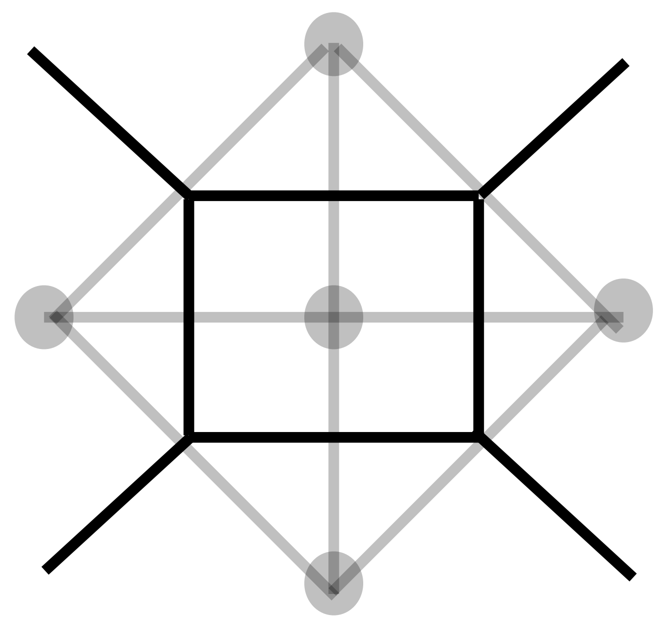
Topological string partition function reproduces Nekrasov partition function for 5d N=1 gauge theory on S^1

Relation between 5-brane web and toric diagram

5-brane web



(Dual graph of) toric diagram for toric Calabi-Yau 3-fold



They are "equivalent" ['97 Leung-Vafa]
and actually connected by the chain of string theory duality ['98 Karch, Lust, Smith]

Find the corresponding M5-brane configuration for given 5-brane web diagram

Equivalent problem

Find local mirror of given toric Calabi-Yau 3-fold

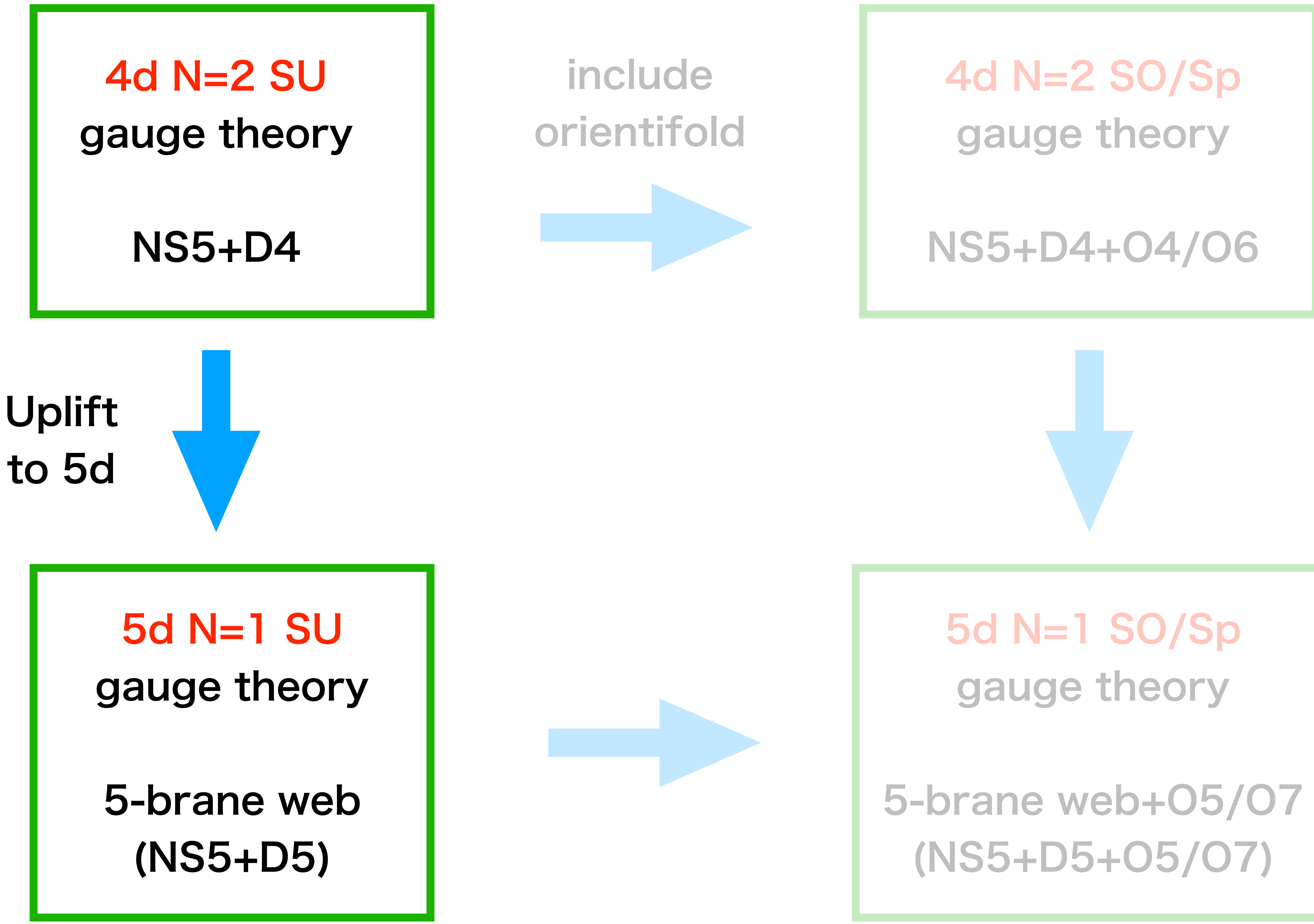
Find the corresponding M5-brane configuration for given 5-brane web diagram with orientifold plane

Equivalent problem ?

Find local mirror of given non-toric Calabi-Yau 3-fold (“Generalization” of the resolution of D-type singularity?)

Unlike the toric case, the correspondence is not established enough although there are various examples to support it.

Generalization



Generalization

4d N=2 SU
gauge theory

NS5+D4

include
orientifold

4d N=2 SO/Sp
gauge theory

NS5+D4+O4/O6

Uplift
to 5d

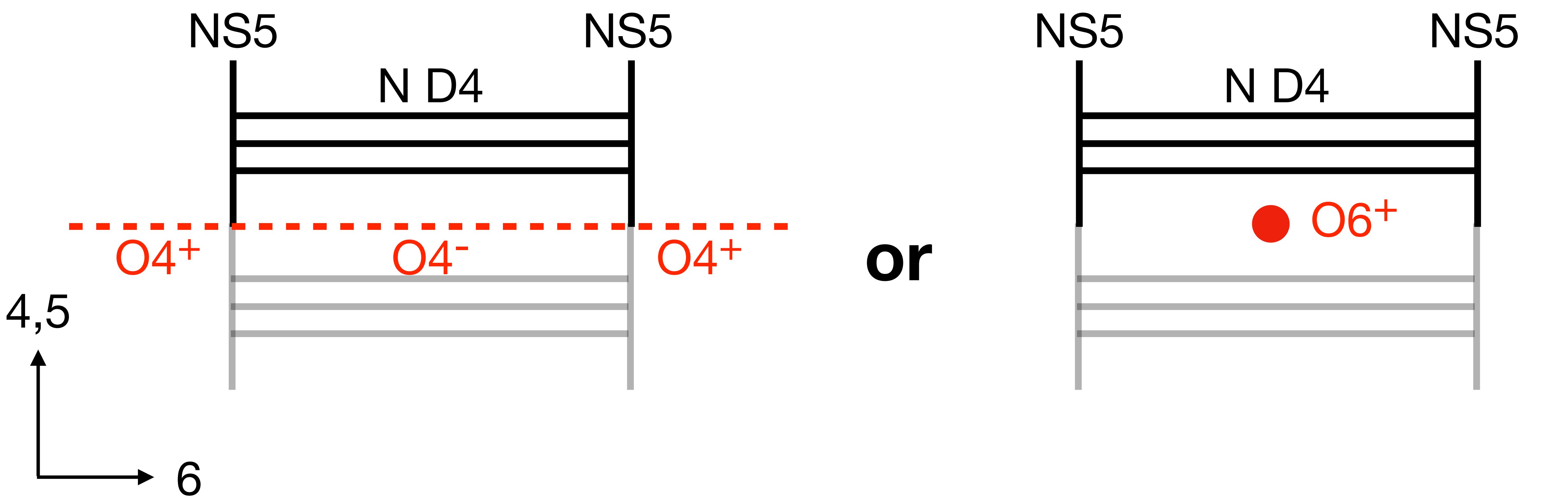
5d N=1 SU
gauge theory

5-brane web
(NS5+D5)

5d N=1 SO/Sp
gauge theory

5-brane web+O5/O7
(NS5+D5+O5/O7)

Brane setup for 4d N=2 SO(2N)



	0	1	2	3	4	5	6	7	8	9
NS5-brane	—	—	—	—	—	—	•	•	•	•
D4-brane	—	—	—	—	•	•	—	•	•	•
O4-plane	—	—	—	—	•	•	—	•	•	•
O6-plane	—	—	—	—	•	•	•	—	—	—

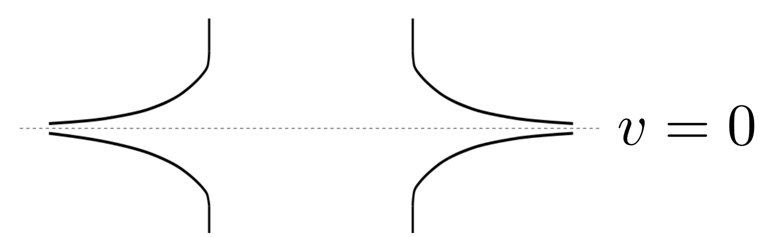
SW curve for 4d N=2 SO(2N)

We need to impose the following constraints:

		O4		O6
①	**Invariant under**	$(v,t) \to (-v,t)$	**or**	$(v,t) \to (-v,t^{-1})$

$$v = x_5 + ix_4, \qquad t = e^{-\frac{x_6 + ix_{11}}{R_M}}$$

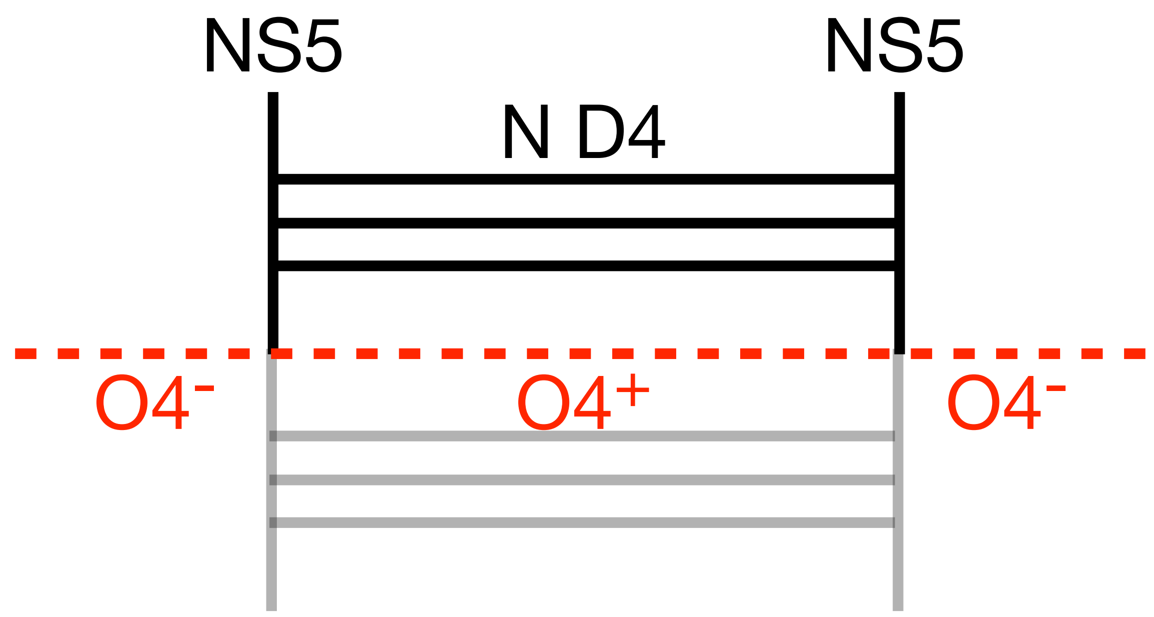
② $v \to 0 \text{ as } t \to 0, \infty$



$$\Rightarrow \quad v^2t^2 + \underline{p(v^2)}t + v^2 = 0$$

Polynomial of v²

SW curve for 4d N=2 Sp(N)



There was a confusion

$$t^2 + v^2 p(v^2) t + 1 = 0$$

VS

$$t^2 + (v^2 p(v^2) \underline{+ 2}) t + 1 = 0$$

['97 Brandhuber, Sonnenschein, Theisen, Yankielowicz]

(cf ['97 Marshakov, Mironov, Morozov])

['97 Landsteiner, Lopez, Lowe]

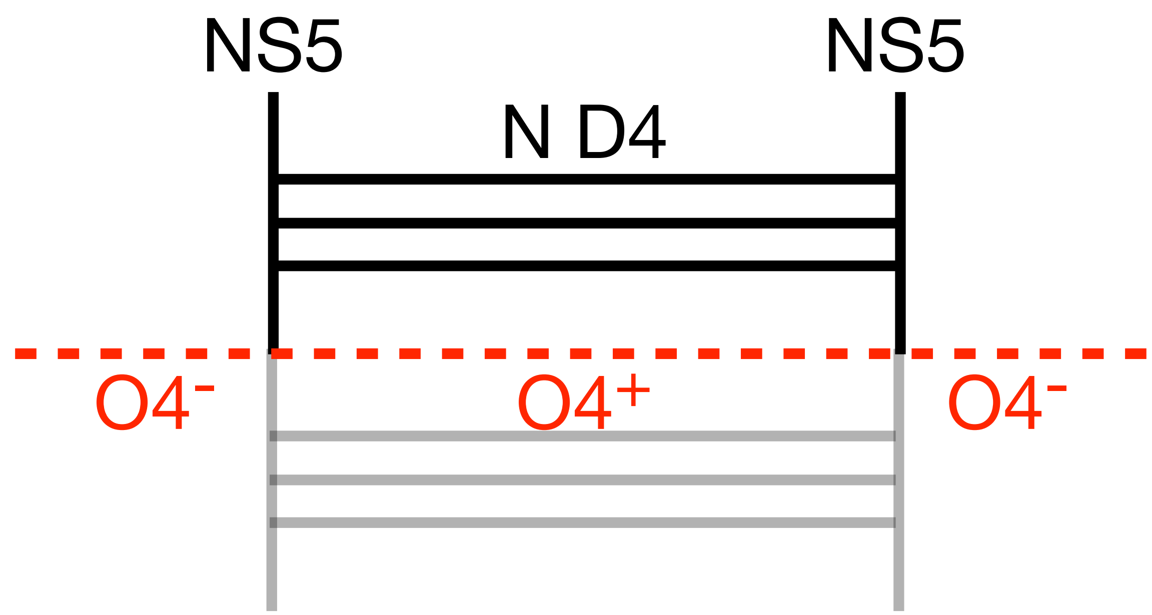
(cf ['95 Martinec, Warner] , ['96 Argyres, Shapere])

v^2 corresponding to $O4^-$

Double root at $v = 0$

$$(t^2 + 2t + 1 = 0)$$

SW curve for 4d N=2 Sp(N)



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v^2 corresponding to $O4^-$

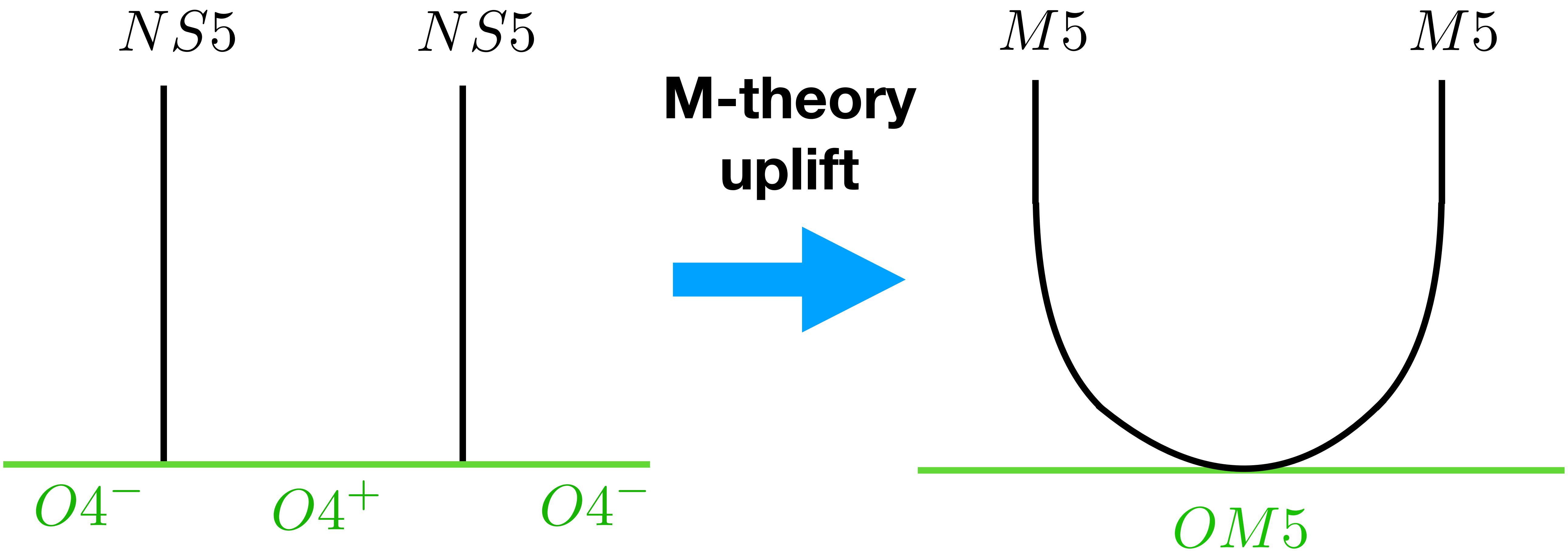
Double root at $v = 0$

$(t^2 + 2t + 1 = 0)$

Boundary condition at OM5-plane

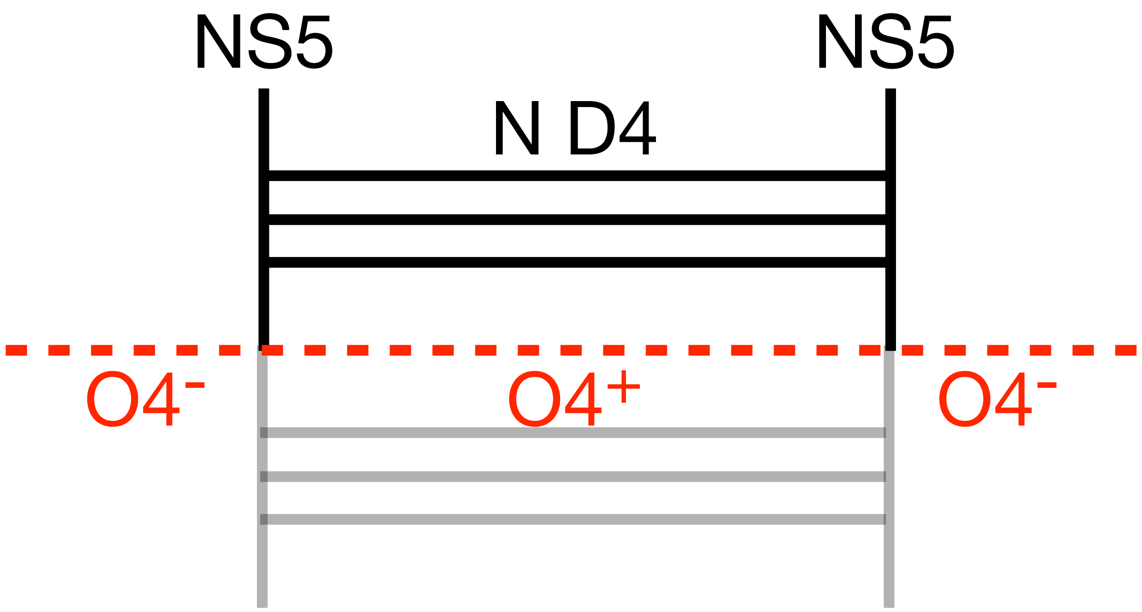
The curve has a double root at OM5-plane

cf [Hori ’96]
[Landsteiner, Lopez, Lowe 97’]

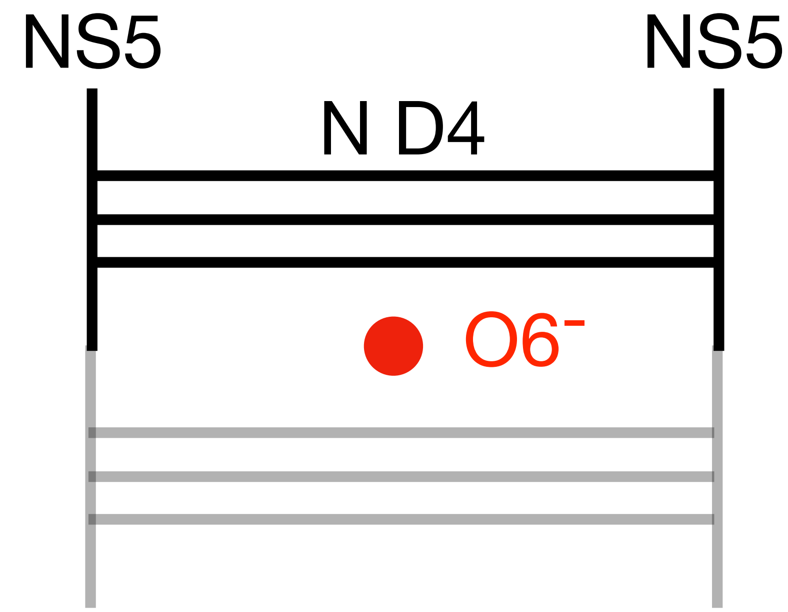


$\text{Double root at } v = 0$

SW curve for 4d N=2 Sp(N)



or



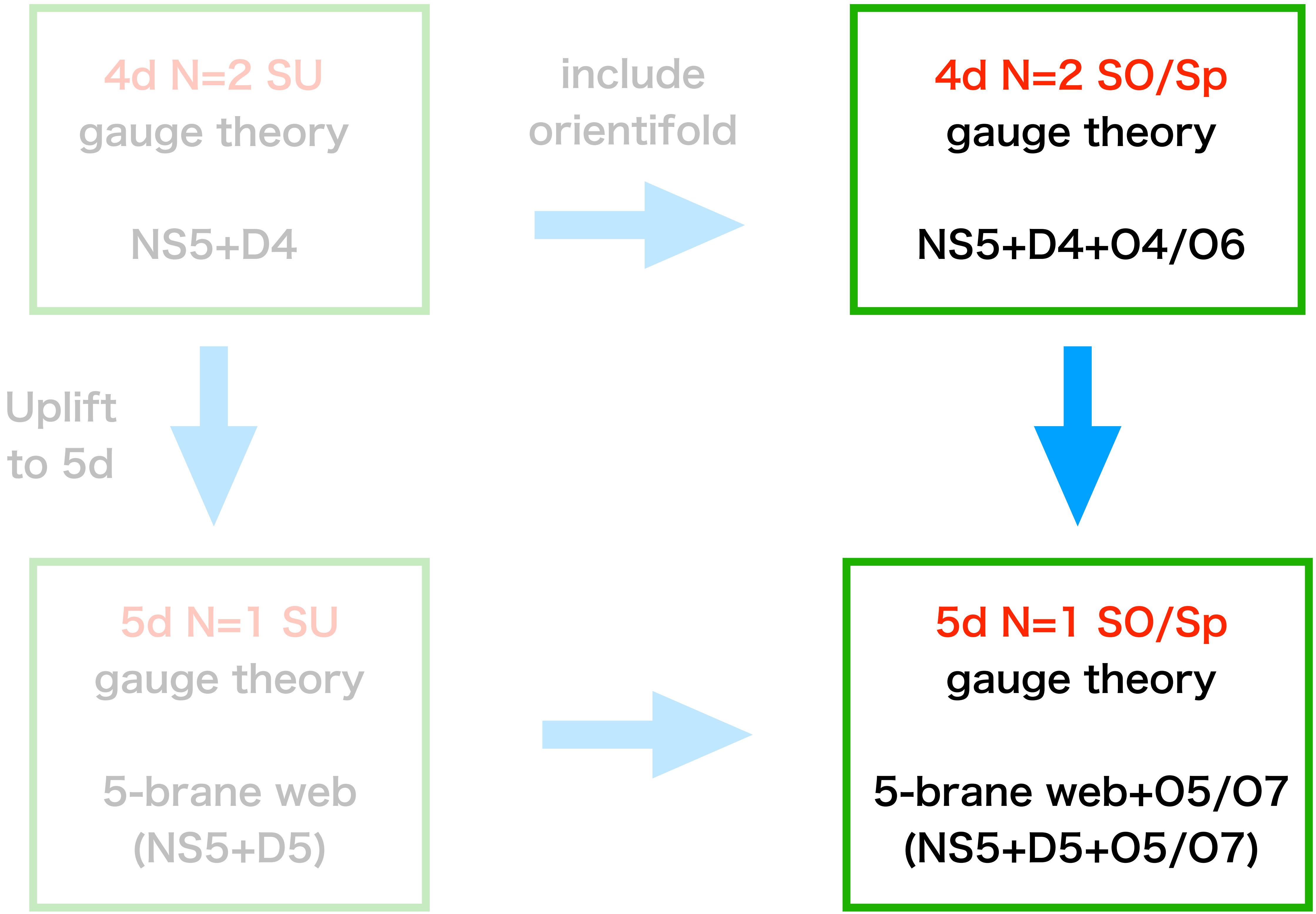
['97 Landsteiner, Lopez, Lowe]

['97 Landsteiner, Lopez]

$$t^2 + (v^2 p(v^2) + 2)t + 1 = 0$$

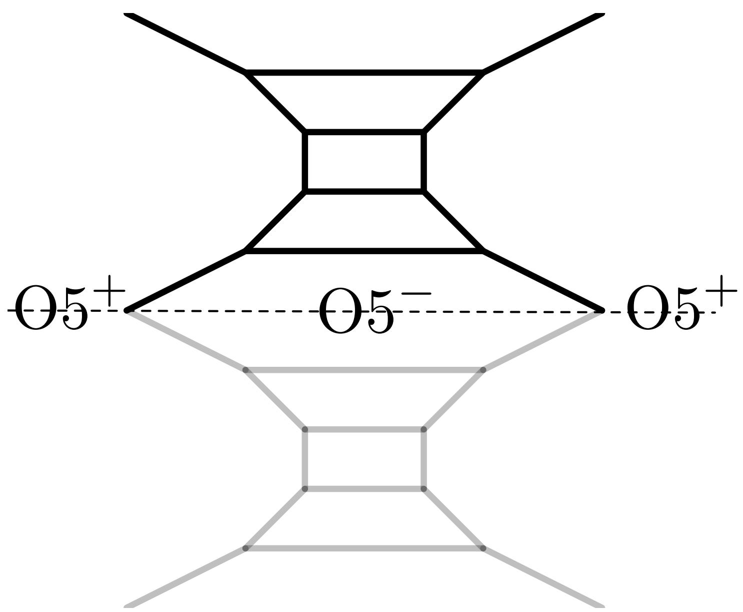
Seiberg-Witten curves
from
5-brane webs with orientifold planes

Generalization



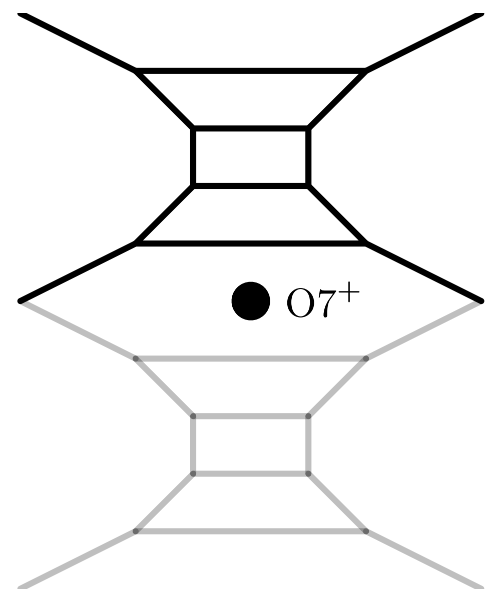
SW curve for 5d N=1 SO(2N) on S^1

['97 Brandhuber, Itzhaki, Sonnenschein, Theisen, Yankielowicz]

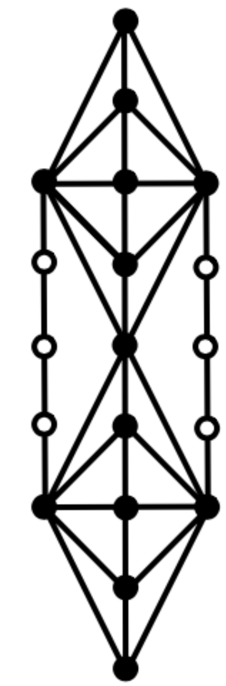


or

['23 Hayashi, Kim, Lee, Yagi]



Dual graph



① **Invariant under** $(w,t) \to (w^{-1},t)$ **or** $(w,t) \to (w^{-1},t^{-1})$

② $w \to \pm 1$ as $t \to 0$ and $t \to \infty$

(O5 $\to$ Two O4s at $w = \pm 1$, O7 $\to$ Two O6s at $(w,t) = (\pm 1, t)$)

$$(w - w^{-1})^2 t + \underline{p(w + w^{-1})} + (w - w^{-1})^2 t^{-1} = 0$$

Polynomial of $w+w^{-1}$

SW curve for 5d N=1 Sp(N) on S^1

4d N=2

$$t^2 + v^2 p(v^2) t + 1 = 0$$

['97 Brandhuber, Sonnenschein, Theisen, Yankielowicz]

(cf ['97 Marshakov, Mironov, Morozov])

VS

$$t^2 + (v^2 p(v^2) \underline{+ 2}) t + 1 = 0$$

['97 Landsteiner, Lopez, Lowe]

(cf ['95 Martinec, Warner] , ['96 Argyres, Shapere])

5d uplift

5d N=1

$$t^2 + (w - w^{-1})^2 p(w + w^{-1}) t + 1 = 0$$

['97 Brandhuber, Itzhaki, Sonnenschein, Theisen, Yankielowicz]

?

SW curve for 5d N=1 Sp(N) on S¹

4d N=2

$$t^2 + v^2 p(v^2) t + 1 = 0$$

['97 Brandhuber, Sonnenschein, Theisen, Yankielowicz]

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5d uplift

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['17 H.Hayashi, S-S. Kim, K. Lee, F.Y.] (Sp(1) from O5)

['21 X.Li, F.Y.] (Sp(N) from O5)

['23 H.Hayashi, S-S. Kim, K. Lee, F.Y.] (Sp(N) from O7)

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Inconsistent/Consistent with Nekrasov partition function obtained by blow up formula. ['21 Brini-Osuga]

SW curve for 5d N=1 Sp(N) on S¹

4d N=2

$$t^2 + v^2 p(v^2) t + 1 = 0$$

['97 Brandhuber, Sonnenschein, Theisen, Yankielowicz]

(cf ['97 Marshakov, Mironov, Morozov])

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5d uplift

5d N=1

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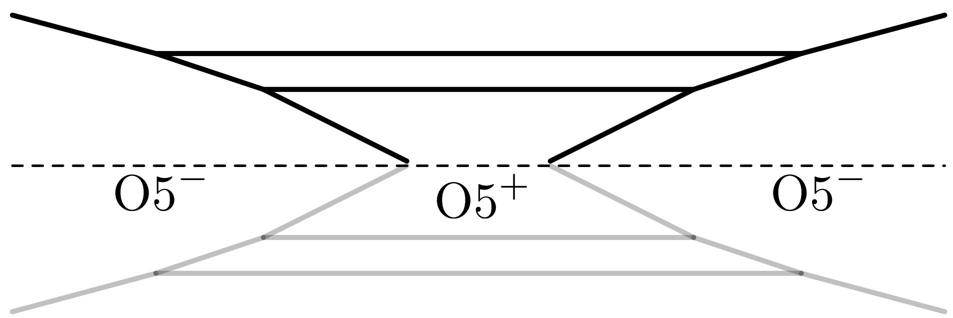
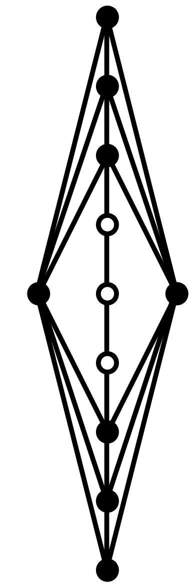




① **Invariant under** $(w, t) \to (w^{-1}, t)$

② **Double root at** $w = \pm 1$

$$t^2 - \left(2 + (w - w^{-1})^2 p(w + w^{-1})\right) t + 1 = 0$$

$(t = 1 \text{ at } w \pm 1)$

$\Rightarrow (N, \theta) = (\text{odd}, 0), (\text{even}, \pi)$

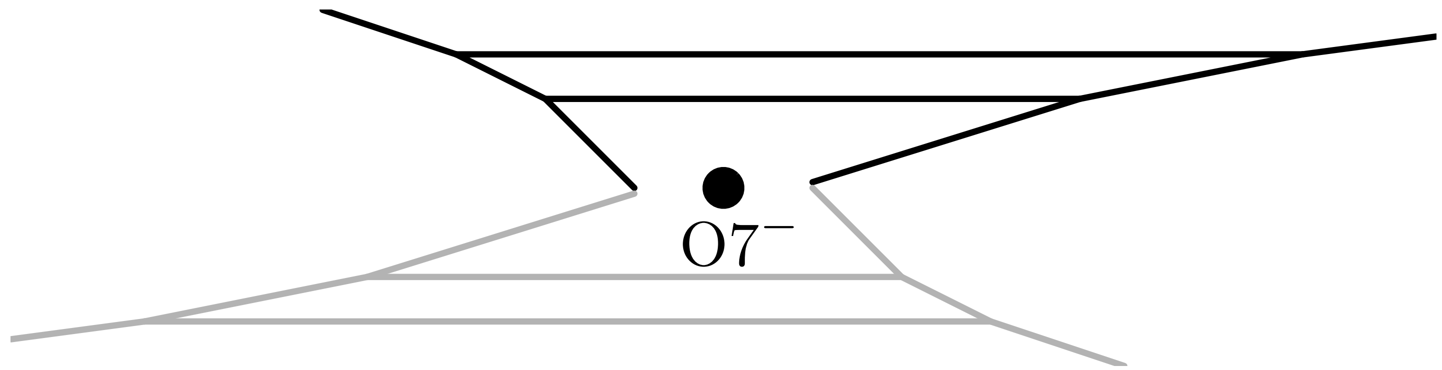
$$t^2 - \left(w + w^{-1} + (w - w^{-1})^2 p(w + w^{-1})\right) t + 1 = 0$$

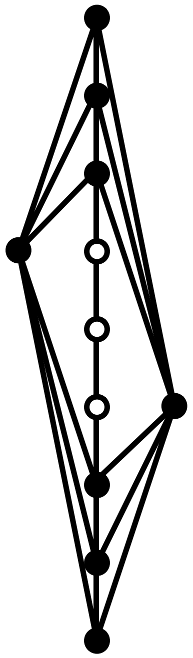
$(t = 1 \text{ at } w = 1, \quad t = -1 \text{ at } w = -1)$

$\Rightarrow (N, \theta) = (\text{even}, 0), (\text{odd}, \pi)$

θ: discrete theta angel

SW curve for 5d N=1 Sp(N) on S¹ from O7





① **Invariant under** $(w,t) \to (w^{-1}, t^{-1})$

② **Double root** $t=1$ **at** $w = \pm 1$ (O7 $\to$ Two O6s at $(w,t) = (\pm 1, t)$)

$$w^{2n} t - \left(2 + (w - w^{-1})^2 p(w)\right) + w^{-2n} t^{-1} = 0$$

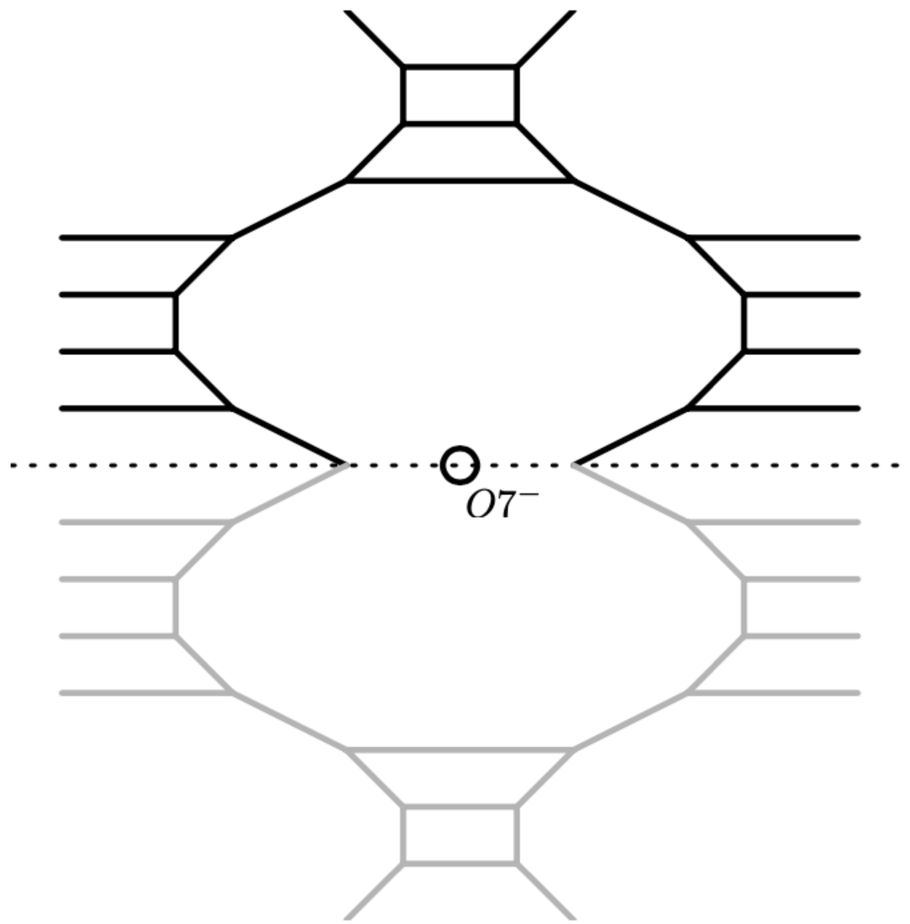
➡ $(N, \theta) = (\text{odd}, 0), (\text{even}, \pi)$

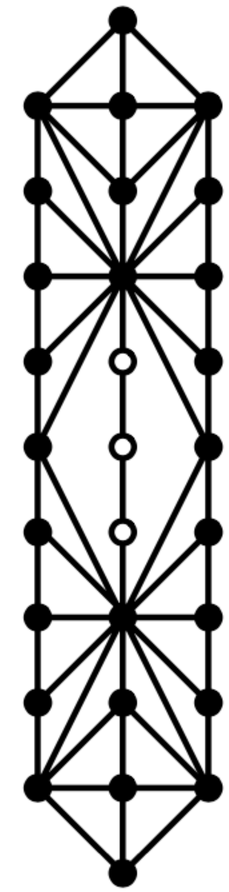
$$w^{2n-1} t - \left(w + w^{-1} + (w - w^{-1})^2 p(w + w^{-1})\right) + w^{-2n+1} t^{-1} = 0$$

➡ $(N, \theta) = (\text{even}, 0), (\text{odd}, \pi)$

Agree with the previous results by $t \to w^{-2n} t, \quad t \to w^{-2n+1} t,$

Adding flavor to 5d N=1 Sp(N) on S^1 from O7

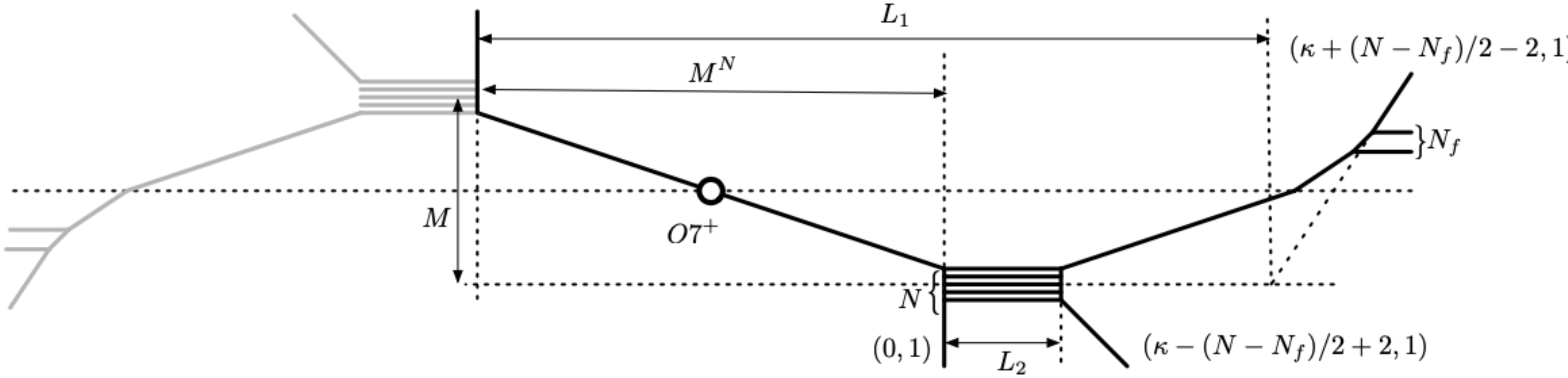


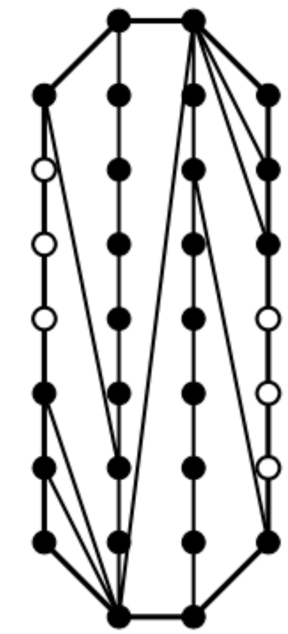


$$p(w)t + \left(-\frac{p(1)(w+1)^2}{2w} + \frac{p(-1)(w-1)^2}{2w} + (w - w^{-1})^2 \hat{p}(w)\right) + p(w^{-1})t^{-1} = 0$$

$$p(w) = w^{-\frac{N_f}{2} - \kappa} \prod_{i=1}^{N_f} (w - M_i), \qquad M_i = e^{-\beta m_i}$$

SW curve for 5d N=1 SU(N)+1Sym+Nf F from O7+



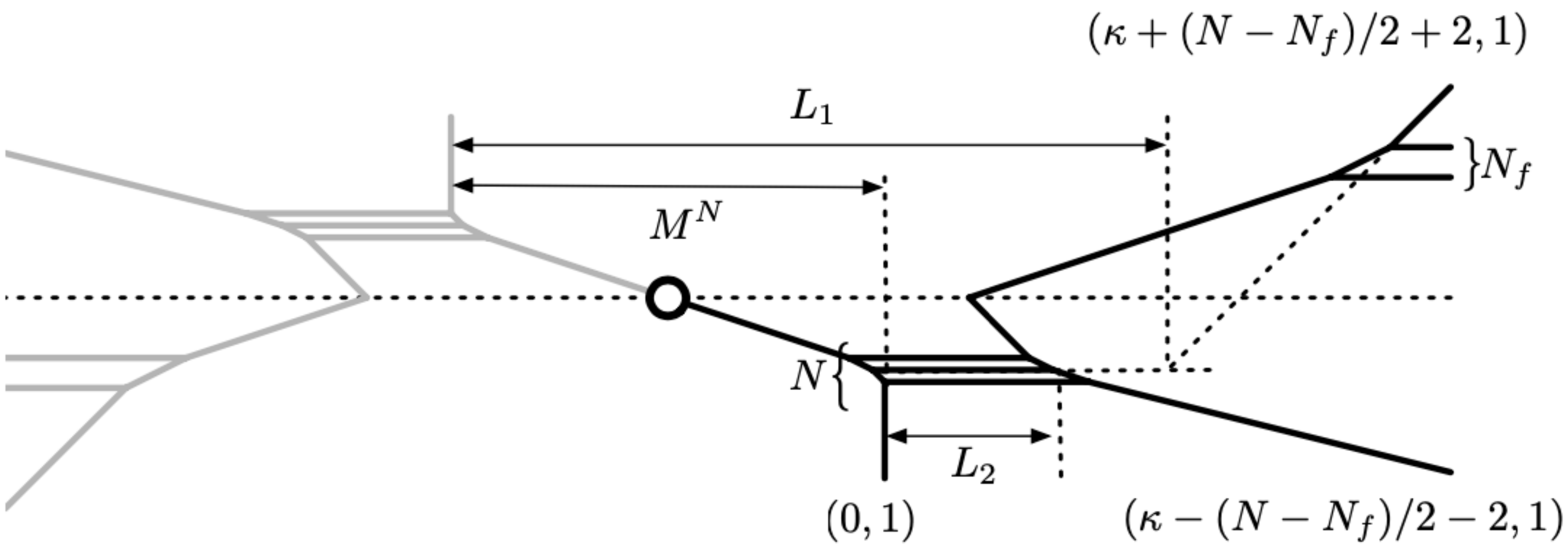


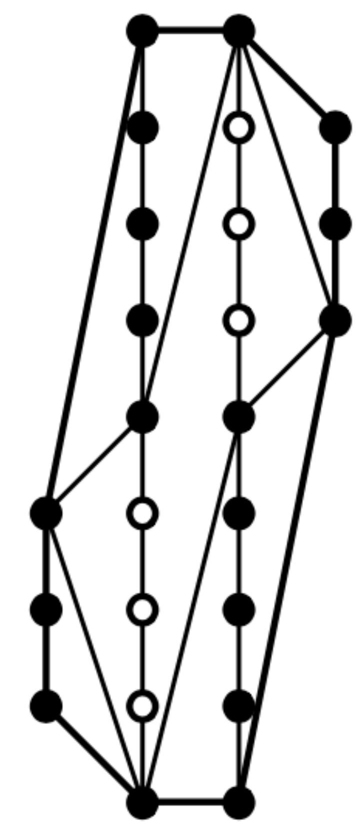
① **Invariant under** $(w,t)\rightarrow(w^{-1},t^{-1})$

② $w\rightarrow\pm1$ as $t\rightarrow0$ and $t\rightarrow\infty$

$$(w-w^{-1})^2\,\check{p}_3(w)t^3+p_2(w)\,t^2-p_2(w^{-1})\,t-(w-w^{-1})^2\,\check{p}_3(w^{-1})\;=0\,.$$

SW curve for 5d N=1 SU(N)+1AS+Nf F from O7⁻





① **Invariant under** $(w,t)\to(w^{-1},t^{-1})$

② **Triple root** $t=1$ **at** $w=\pm 1$

③ $\left|\dfrac{\partial t}{\partial w}(w=\pm 1)\right| < \infty$

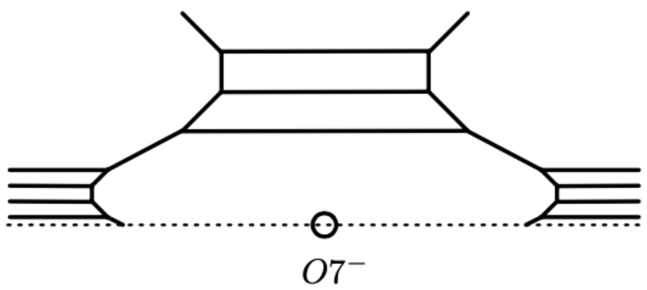
$$p_3(w)t^3+p_2(w)t^2-p_2(w^{-1})t-p_3(w^{-1})=0.$$

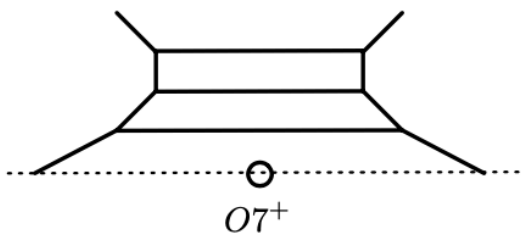
$$p_2(w)=-p_3(w)-\frac{1}{2}p_3(1)\frac{(w+1)^2}{w}+\frac{1}{2}p_3(-1)\frac{(w-1)^2}{w}+(w-w^{-1})^2\hat{p}_2(w)$$

Observation

SW curve for 5d N=1 Sp(N) + 8F → **SW curve for 5d N=1 SO(2N)**

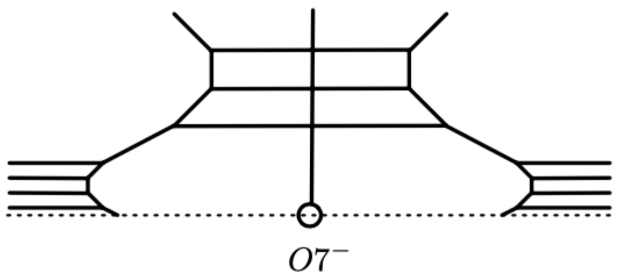
$$M_1 = M_2 = M_3 = M_4 = 1,$$
$$M_5 = M_6 = M_7 = M_8 = -1,$$

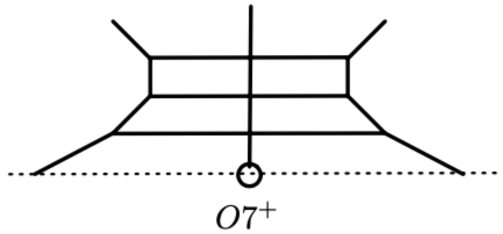




SW curve for 5d N=1 SU(N) + 1AS + 8F → **SW curve for 5d N=1 SU(N) + 1Sym**

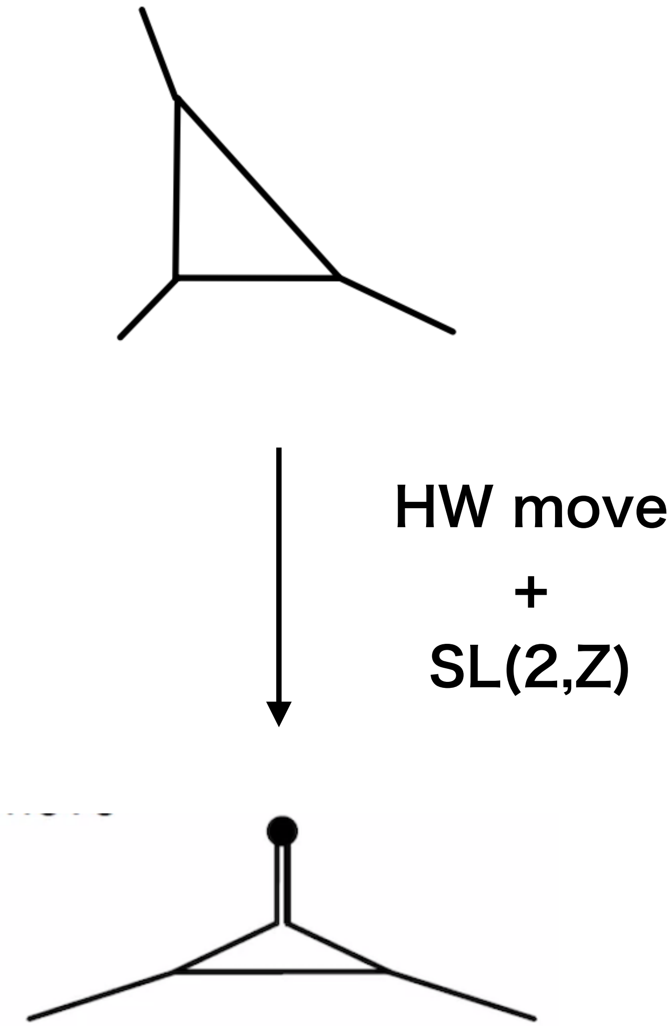
$$M_1 = M_2 = M_3 = M_4 = 1,$$
$$M_5 = M_6 = M_7 = M_8 = -1,$$

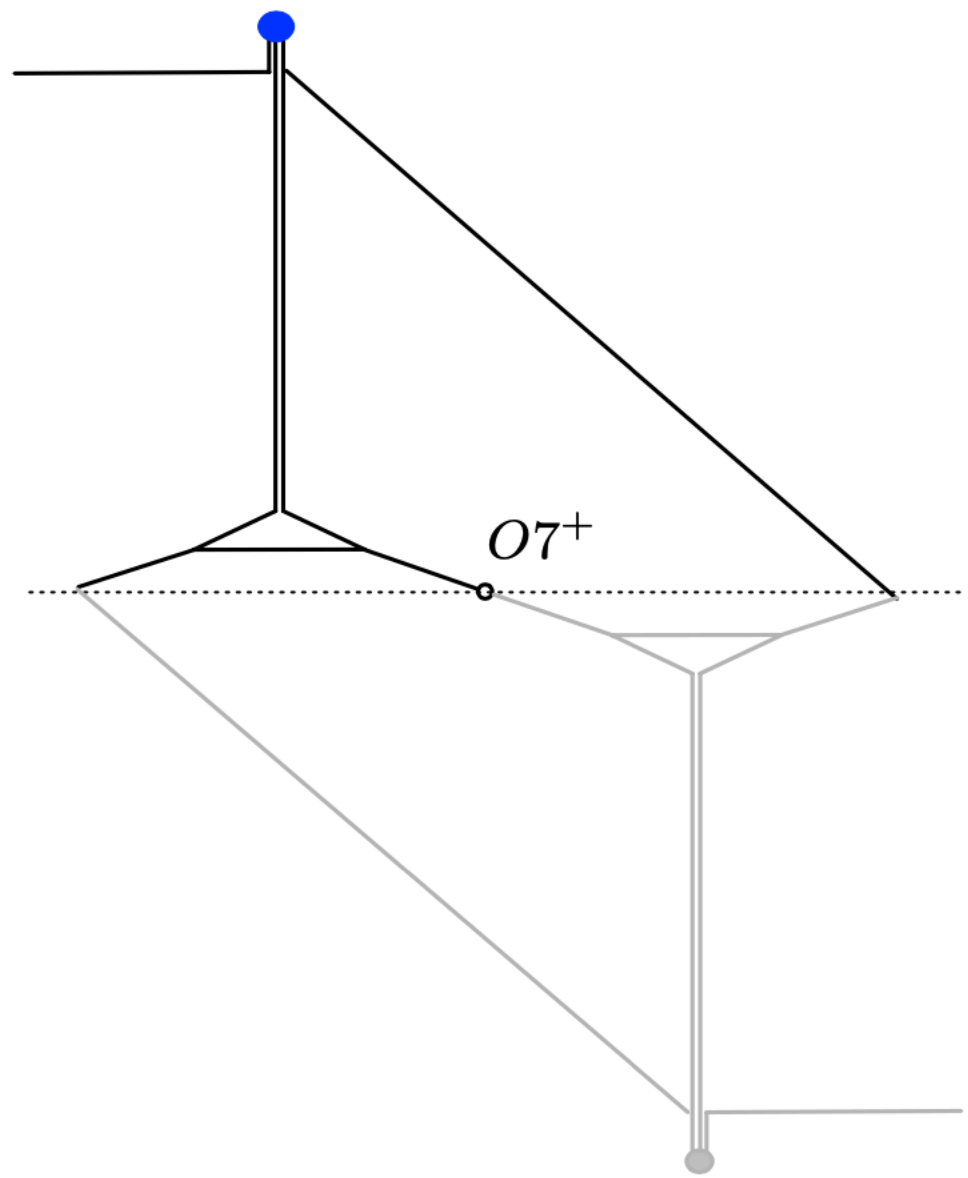


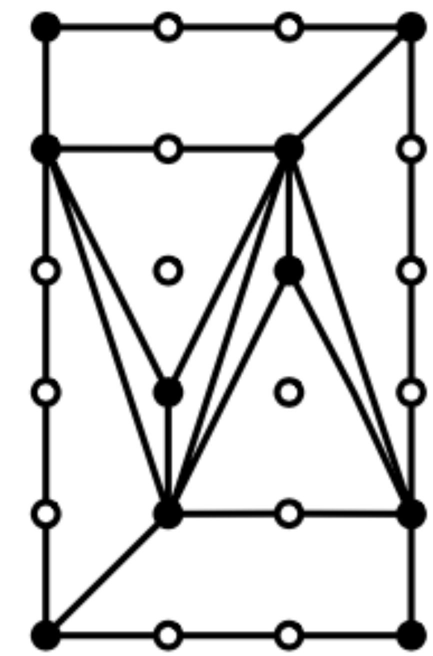


Interpretation: "$O7^+ = O7^- + 8$ D7"

Non-Lagrangian theory realized by 5-brane web with O7







Local P^2 theory
(decoupling limit of $SU(2)_\pi$ SYM)

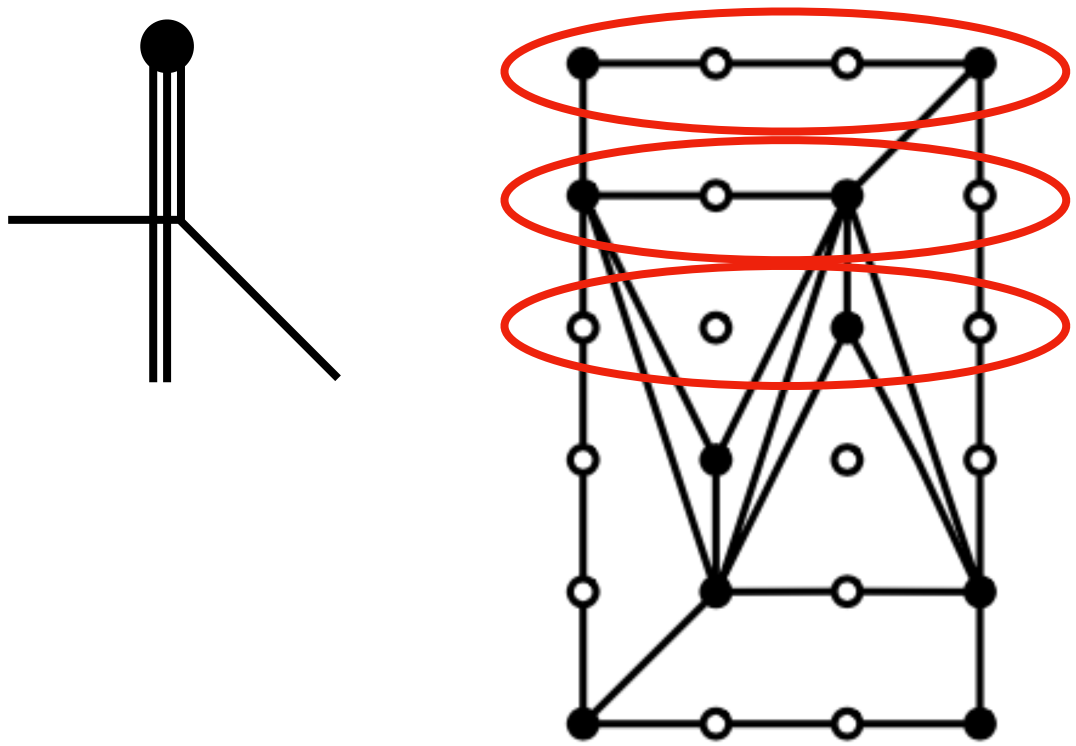
Local P^2 theory
+ “adjoint”

SW curve for local P² theory + "adjoint"

① **Invariant under** $(w,t)\to(w^{-1},t^{-1})$

② $w\to\pm1$ as $t\to 0$ and $t\to\infty$

③



$\propto (t-M)^3$

$\propto (t-M)^2(t-c_0)$

$\propto (t-M)(t-c_1)(t-c_2)$

$$
\begin{aligned}
&t^3\left(w^2-1\right)^2\left(w-M^3\right)\\
&+t^2\Big(-3Mw^5+(2M^4+M^{-2})w^4+Uw^3-MUw^2-(M^5+2M^{-1})w+3M^2\Big)\\
&+t\Big(-3M+(2M^4+M^{-2})w+Uw^2-MUw^3-(M^5+2M^{-1})w^4+3M^2w^5\Big)\\
&+\left(1-w^2\right)^2\left(1-M^3w\right)=0\ .
\end{aligned}
$$

Observation

Double discriminant of SW curve for local P^2 theory + "adjoint"

$$\Delta_{O7^+} = \Delta_{\rm phys}\Delta_{\rm unphys}$$

$$\begin{aligned}\Delta_{\rm phys}(U) = &\left(\frac{U^2}{M^2} - 2(\chi_1+4)\frac{U}{M} - \chi_1^3 + 3\chi_1^2 + 12\chi_1 + 8\right)^5 \\ &\times\left(\frac{U^3}{M^3} + (15\chi_1 - 32)\frac{U^2}{M^2} + (3\chi_1^2 - 32\chi_1 - 32)\frac{U}{M}\right. \\ &\left. - 27\chi_1^4 - 19\chi_1^3 + 120\chi_1^2 + 192\chi_1 + 80\right),\end{aligned}$$

$$\begin{aligned}\Delta_{\rm unphys}(U) = &\, 256\, M^{36}(\chi_1-2)^3 \\ &\times\left(\frac{U^3}{M^3} - 3(\chi_1^2 - 4\chi_1 + 10)\frac{U^2}{M^2} + 3(\chi_1^4 - 17\chi_1^3 + 45\chi_1^2 - 8\chi_1 - 8)\frac{U}{M}\right. \\ &\left. - \chi_1^6 + 12\chi_1^5 + 57\chi_1^4 - 236\chi_1^3 + 30\chi_1^2 + 120\chi_1 + 80\right)^3 .\end{aligned}$$

$$\chi_1 = M^2 + M^{-2}$$

$$\Delta_{\rm phys}(U) = 0 \quad \leftrightarrow \quad \Delta_{\rm frozen}(u) = 0$$

$$U = M(u - 56 - \chi_1)$$

Double discriminant of SW curve for 5d Sp(1) + 7F with tuned masses

$$\mathsf{M}_0 = \widetilde{M}^{-1}, \qquad \mathsf{M}_{1,2,3} = \widetilde{M}, \qquad \mathsf{M}_{4,5,6,7} = -\widetilde{M}$$

$$\begin{aligned}\Delta_{\rm frozen}(u) &= g_2^3 - 27g_3^2 \\ &= \left(u^2 - 4(\chi_1+30)u - \chi_1^3 + 6\chi_1^2 + 244\chi_1 + 3592\right)^4 \\ &\quad\times\left(u^3 + 12u^2\chi_1 - 200u^2 - 24u\chi_1^2 - 1312u\chi_1 + 12960u\right. \\ &\quad\left. - 27\chi_1^4 - 8\chi_1^3 + 1464\chi_1^2 + 36064\chi_1 - 274096\right) ,\end{aligned}$$

$$\chi_1 = \widetilde{M}^2 + \widetilde{M}^{-2}$$

Conclusion

For 5-brane webs with O-planes,
the SW curve is obtained by imposing
Z_2 invariance + further constraint

Non-trivial relation between two SW curves is obtained by "$O7^+ \Leftrightarrow O7^- + 8D8$"

This technique is applicable for non-Lagrangian theory like local P^2 theory + "adjoint"