Schur index and Modularity

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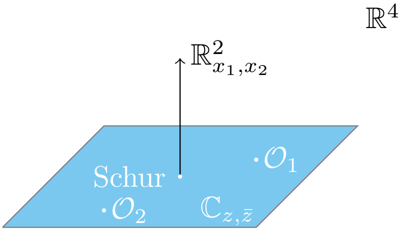
Outline

- Introduction: review 4d/2d (SCFT/VOA) correspondence and motivation
- Schur index and non-local operators
- Modular property of index
- Modular differential equations and (quasi-)modularity

Introduction

- 4d/2d (SCFT/VOA) correspondence [Beem, Lemos, Liendo, Peelaers, Rastelli, Rees]:

4d $\mathcal{N} = 2$ SCFT $\mathcal{T}$ $\xrightarrow{\mathbb{V}}$ 2d vertex operator algebra / chiral algebra (VOA) $\mathbb{V}(\mathcal{T})$



Associated VOA: a review

- Gauge-inv. **Schur ops** on the $\mathbb{C}_{z,\bar{z}}$: form associated $\mathbb{V}(\mathcal{T})$
  - As holomorphic **cohomology class** of $\mathbb{Q}$: $\mathcal{O}(z) = [\mathcal{O}(z,\bar{z})]_{\mathbb{Q}}$
  - Holomorphic OPE coefficients between $\mathcal{O}(z)$
  - Schur condition

$$E - 2R - j_1 - j_2 = 0, \qquad r + j_1 - j_2 = 0 \ . \tag{1}$$

- 4d $\mathcal{R}$-symmetry $\to$ Virasoro subalgebra $c_{\text{2d}} = -12c_{\text{4d}} < 0$

$$(J_{\mathcal{R}})^{11}_{+\dot{+}} \to T \tag{2}$$

- 4d $\mathfrak{f}$ flavor-symmetry $\to \widehat{\mathfrak{f}}_{k_{\text{2d}}}$ subalgebra, $k_{\text{2d}} = -\frac{1}{2}k_{\text{4d}}$,

$$(M^{11})^A \to J^A \ . \tag{3}$$

- Many aspects in the area, to list a few
  Class-$\mathcal{S}$ and T_N: [Beem, Peelaers, Rastelli, van Rees][Lemos, Peelaers][Kiyoshige, Nishinaka] ...
  Argyres-Douglas: [Song, Xie, Yan] [Xie, Yan, Yau] [Dedushenko, Wang] [Buican, Nishinaka] [Kozcaz, Shakirov, Yan][Creutzig][Kang, Laurie, Lee, Song] ...
  MDE, defects, modules: [Gang, Koh, Lee][Cordova, Gaiotto, Shao][Bianchi, Lemos][Nishinaka, Sasa, Zhu][Beem, Rastelli][YP, Wang, Zheng][Hatsuda, Okazaki][Kaidi, et.al.][Arakawa][Li, Li, Yan]...
  Free field realization: [Adamovic][Beem, Meneghelli, Rastelli][Bonetti, Meneghelli, Rastelli] ...

Associated VOA: Schur index

- Schur ops: **counted** by the **Schur index** [Gadde, et.al.],

$$\mathcal{I}_{\mathcal{T}} \equiv \mathrm{str}_{\mathcal{H}}\, q^{E-R+\frac{c_{4\mathrm{d}}}{2}} \mathbf{b}^{\mathbf{f}} = \mathrm{str}_{\mathbb{V}[\mathcal{T}]}\, q^{L_0-\frac{c_{2\mathrm{d}}}{24}} \mathbf{b}^{\mathbf{f}} \ , \qquad (4)$$

  where $q \equiv e^{2\pi i\tau}$, $\mathbf{b}, \mathbf{f}$ are flavor fugacities and Cartan gen's.

- Special limit of superconformal index
- Key equality:

  Schur index $\mathcal{I}_{\mathcal{T}}$ = **vacuum** module character of $\mathbb{V}[\mathcal{T}]$

Associated VOA: Schur index

Computing Schur indices (focus on Lagrangian theories):

- Direct counting Schur opeerators or identifying the VOA [Gadde, Rastelli, Razamat, Yan]: a series expansion
- From **2d q-Yang-Mills** partition functions [Gadde, Rastelli, Razamat, Yan]: an infinite sum over representations
- From **localization** on $S^3 \times S^1$, or zero-coupling limit (independence of g_{YM}) [Gadde, et.al.][YP, Peelaers][Dedushenko, Fluder][Jeong]: multivariate **contour integral** formula

$$\mathcal{I}_{\text{Lagrangian}} = \oint_{|a|=1} \left[\frac{da}{2\pi i a}\right]_{\text{Haar}} \mathcal{Z}(a) \qquad (5)$$

Also compute **Schur correlators** on $S^3 \times S^1$ [YP, Peelaers]

- Contour integral formula:
  $\mathcal{Z}(a)$ counts Schur operators candidates (in the zero coupling limit)
  the Haar measure integration enforces **gauge-invariance**
- Exact computation in closed-form: [Bourdier, Drukker,Felix][YP, Peelaers][Beem, Sinh, Razamat][Huang][Hatsuda, Okazaki][Du, Huang, Wang] ...

Associated VOA: modules

- VOAs are interesting objects from representation-theoretic perspective
- Rational VOA $\mathbb{V}$
  - Finitely many irreducible $\mathbb{V}$-modules M_j
  - Modularity of characters ch_j

$$\mathrm{ch}_i \xrightarrow{\Gamma} \sum_j \rho(\Gamma)_{ij}\, \mathrm{ch}_j \tag{6}$$

    and modular inv. partition function $Z \sim \sum_{i,j} \mathcal{M}_{ij}\, \mathrm{ch}_j\, \overline{\mathrm{ch}_j}$
  - Verlinde formula

$$N_{ij}^k = \sum \frac{S_{i\ell} S_{j\ell} S^*_{\ell k}}{S_{0\ell}} \tag{7}$$

Associated VOA: modules

- SCFT/VOA correspondence provides a huge set of chiral algebras: $\mathbb{V}[\mathcal{T}]$, **non-unitary**, **non-rational** in general [Beem, et.al.]
- Representation theory under exploration [Kac, Wakimoto][Arakawa][Adamović][Cordova, Gaiotto, Shao][Xie, Yan][Li, Li, Yan]...
- Simplest module
  - **Vacuum** module $M_0 \sim \mathbb{V}$
  - **Vacuum** character $\mathrm{ch}_0 = \mathcal{I}_{\mathcal{T}}$
- Sources of **non-vacuum** modules from 4d physics:

  - superconformal surface defects in $\mathcal{T}$ [Cordova, Gaiotto, Shao][Nishinaka, Sasa, Zhu][Bianchi, Lemos][Beem, Rastelli][Beem, Peelaers]

  - superconformal line operators [Gang, Koh, Lee][Cordova, Gaiotto, Shao]

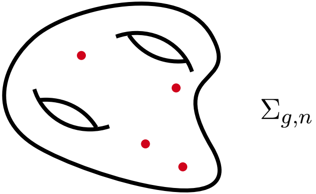
Goal

- Explore non-vacuum module characters for simple theories
- Study modular property of the their index
- Study the modularity of flavored modular linear differential equations (FMLDE)
- Use the modularity of FMLDE to constrain vertex operator algebra

Index and FMLDEs

Schur index

- Schur index with or without **non-local operators**
- General belief and known examples
  - Schur index **without** non-local operators: vacuum character
  - Schur index **with** non-local operators: non-vacuum characters
- Helpful tools
  - New exact integration formula
  - Flavored modular linear differential equations (FMLDE)
- Simplest playground: A_1 theories of class-$\mathcal{S}$ and 4d $\mathcal{N} = 4$ $SU(N)$ SYM



Tools: new integration formula

- Schur index (with/without non-local operators) of Lagrangian theories: nested contour integral

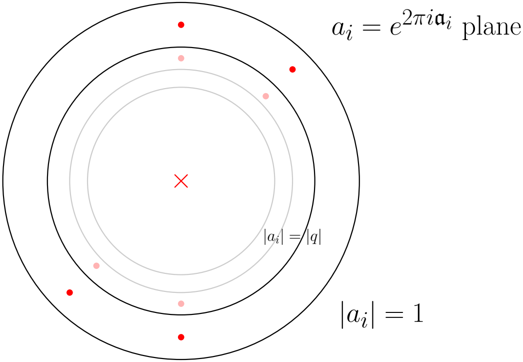
$$\operatorname{Res} \oint_{|a_i|=1} \underbrace{\left[\frac{d\mathbf{a}}{2\pi i\mathbf{a}}\right]}_{\text{Haar}} \underbrace{R(\mathbf{a})}_{\text{poly.}} \underbrace{f(\mathbf{a},\mathbf{b})}_{\text{elliptic}} \underbrace{E_{k_1}\begin{bmatrix}\pm 1\\ \mathbf{ab}\end{bmatrix} E_{k_2}\begin{bmatrix}\pm 1\\ \mathbf{ab}\end{bmatrix}(\cdots)}_{\text{twisted Eisenstein}} \ .$$

Tools: new integration formula

- New **integration formula** [YP, Peelaers][Zheng, YP, Wang]

Extract "residue" from **non-isolated** singularity

(determined by the converging series of isolated singularities)



- **Elliptic** function $\left(f(\mathfrak{a}) = f(\mathfrak{a}+\tau) = f(\mathfrak{a}+1)\right)$ contour integral

$$\oint_{|a|=1} \frac{da}{2\pi i a} f(\mathfrak{a}) = f(\mathfrak{a}_0) + \sum_{\text{real/img } \mathfrak{a}_j} R_j E_1 \begin{bmatrix} -1 \\ a_j q^{\pm\frac{1}{2}}/a_0 \end{bmatrix} . \quad (8)$$

- **Elliptic** times Eisenstein $\left(\frac{1}{2}\frac{y}{\sinh(y/2)} = \sum_{\ell\geq 0} \mathcal{S}_\ell y^\ell\right)$

$$\oint_{|z|=1} \frac{dz}{2\pi i z} f(\mathfrak{z}) E_k \begin{bmatrix} -1 \\ za \end{bmatrix} \quad (9)$$

$$= -\mathcal{S}_k \left(f(\mathfrak{z}_0) + \sum_{\text{real/img} \mathfrak{z}_j} R_j E_1 \begin{bmatrix} -1 \\ \frac{z_j}{z_0} q^{\pm\frac{1}{2}} \end{bmatrix} \right) \quad (10)$$

$$- \sum_{\text{real/img} \mathfrak{z}_j} R_j \sum_{\ell=0}^{k-1} \mathcal{S}_\ell \, E_{k-\ell+1} \begin{bmatrix} 1 \\ z_j a q^{\pm\frac{1}{2}} \end{bmatrix} . \quad (11)$$

- Monomial times one Eisenstein (Eulerian polynomial $\sum_{n=0}^{+\infty} \mathsf{Eu}_n(t)\frac{x^n}{n!} = \frac{t-1}{t-e^{(t-1)x}}$)

$$\oint \frac{dz}{2\pi i z} z^n E_k \begin{bmatrix} 1 \\ za \end{bmatrix} = \frac{1}{(k-1)!}\frac{q^n}{a^n}\frac{\mathsf{Eu}_{k-1}(q^n)}{(1-q^n)^k} \,, \tag{12}$$

- Monomial times two Eisenstein

$$\oint \frac{dz}{2\pi i z} z^n E_{k_1} \begin{bmatrix} +1 \\ z \end{bmatrix} E_{k_2} \begin{bmatrix} +1 \\ za \end{bmatrix} \tag{13}$$

$$= \sum_{\ell=0}^{k_1} \frac{1}{\ell!}\frac{q^n}{a^n}\frac{\mathrm{Eu}_{k_2+\ell-1}(q^n)}{(1-q^n)^{k_2+\ell}} \tag{14}$$

$$\times \left[\frac{(-1)^{k_1-\ell}}{(k_2-1)!} + \frac{\ell! a^n}{(k_1-1)!(k_2-k_1+\ell)!}\right] E_{k_1-\ell} \begin{bmatrix} +1 \\ a \end{bmatrix} .$$

Tools: new integration formula

- Application: exact **closed-form** index for different classes of theories [YP, Peelaers][Zheng, YP, Wang][Hatsuda, Okazaki's result for $\mathcal{N} = 4$ with $SU(N)$, Du, Huang and Wang's result for more general gauge groups]

$$\mathcal{I} = \sum_{\text{finite sum}} \underbrace{R(\mathbf{b}, q)}_{\text{rational}} \frac{\vartheta_i \cdots}{\vartheta_j \cdots} E_{k_1} \begin{bmatrix} \pm 1 \\ \mathbf{b} \end{bmatrix} E_{k_2} \begin{bmatrix} \pm 1 \\ \mathbf{b} \end{bmatrix} (\ldots) \qquad (15)$$

- Extract **candidates** of **module characters** from the above closed-form: flavored modular linear differential equations (FMLDE)

- Modular group $SL(2,\mathbb{Z})$ (or $\Gamma^0(2)$)

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \gamma \cdot \tau := \frac{a\tau + b}{c\tau + d} \ . \tag{16}$$

- **Modular form** of **weight-k** (w.r.t. $SL(2,\mathbb{Z})$) form $\phi(\tau)$

$$\phi\left(\gamma \cdot \tau\right) = (c\tau + d)^k \phi(\tau) \ .$$

- **Vector valued modular form** of weight-k

$$\phi_i\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \rho(\gamma)_{ij}\phi_j(\tau), \qquad \gamma \in SL(2,\mathbb{Z}) \ .$$

Tools: flavored modular differential equations

- Bosonic **Rational** CFT:

  finite number of chiral primaries $\mathcal{O}_i \to \mathrm{ch}_i$

- Characters: $q = e^{2\pi i\tau}$, $b_j = e^{2\pi i \mathfrak{b}_j}$

$$\mathrm{ch}_i \coloneqq \mathrm{tr}_{\mathcal{V}_i}\, q^{L_0 - \frac{c}{24}} \prod_j b_j^{f_j} = \mathrm{tr}_{\mathcal{V}_i}\, q^{L_0 - \frac{c}{24}} \mathbf{b}^{\mathbf{f}} \ . \qquad (17)$$

Unflavoring limit always exists

$$\mathrm{ch}_i(\tau) = \mathrm{ch}_i(\tau, \mathbf{b} \to 1) < \infty \ . \qquad (18)$$

- Modularity: vector valued modular form of weight zero,

$$\mathrm{ch}_i(\gamma \cdot \tau) = \sum_j \rho(\gamma)_{ij}\, \mathrm{ch}_j(\tau) \ , \qquad \gamma \in SL(2, \mathbb{Z}) \ . \qquad (19)$$

- Modular property of $\mathrm{ch}_i \Rightarrow$ **unflavored** MLDE from **Wronskian** argument

$$D_q^{(n)}\,\mathrm{ch}_i(\tau) + \sum_{r=0}^{n-1} \phi_{2r}(\tau) D_q^{(n-r)}\,\mathrm{ch}_i(\tau) = 0\ . \tag{20}$$

where ϕ_{2r} are weight-$2r$ **modular forms**, possibly with poles.

- **Serre derivatives**: maps modular forms to modular forms,

$$D_q^{(n)} := \Big(q\partial_q - (2n-2)E_2\Big) \circ \ldots \circ (q\partial_q - 2E_2) \circ (q\partial_q)\ . \tag{21}$$

- Example: $n = 1$, e.g., $(\widehat{\mathfrak{e}}_8)_1$,

$$D_q^{(1)}\,\mathrm{ch} - \underbrace{\left(\frac{D_q^{(1)}\,\mathrm{ch}}{\mathrm{ch}}\right)}_{\phi_2} \mathrm{ch} = 0\ . \tag{22}$$

Tools: flavored modular differential equations

- Unflavored MLDE: constrain and classify rational unitary CFTs [Chandra, Mukhi; Kaidi, Lin, Parra-Martinez; Das, Gowdigere, Mukhi, Santara; Bae, Duan, Lee, Lee, Sakis] ...
- Non-rational case: no Wronskian argument
  - Characters with **no** unflavoring limit

$$\mathrm{ch}_i(\mathbf{b} \to 1) \to \infty \ . \tag{23}$$

  - Characters with complicated modular property, often with $\log q$ terms
  - ϕ's depends on $\mathbf{b}$: not modular
- Second mechanism to get **FMLDE**: null vectors [Gaberdiel, Keller][Beem, Rastelli][Kaidi, et.al.][Peelaers, YP][YP, Wang, Zheng]

Tools: flavored modular differential equations

- A (untwisted, for simplicity) highest weight module M of a **simple** VOA $\mathbb{V}$
- Any **null state** $\mathcal{N}$ in $\mathbb{V}$: vanishing 1pt function

$$0 = \mathrm{tr}_M (-1)^F o(\mathcal{N}) q^{L_0 - \frac{c}{24}} \mathbf{b}^{\mathbf{f}} \ . \qquad (24)$$

Here $\mathbf{b}^{\mathbf{f}} = \prod_i b_i^{f_i}$

Tools: flavored modular differential equations

- Zhu's recursion formula helps compute 1-pt function
- **Zhu's recursion formula** [Zhu][Krauel, Mason][Beem, Rastelli][Beem, Peelaers][YP, Wang, Zheng]: when $J_0 a = 0$

$$\begin{aligned}
& \operatorname{str}_M o(a_{[-h_a]}b) x^{J_0} q^{L_0} \\
= \ & \operatorname{str}_M o(a_{[-h_a]}|0\rangle) o(b) x^{J_0} q^{L_0} \\
& + \sum_{n=1}^{+\infty} E_{2k} \begin{bmatrix} e^{2\pi i h_a} \\ 1 \end{bmatrix} \operatorname{str}_M o(a_{[-h_a+2k]}b) x^{J_0} q^{L_0} \ .
\end{aligned} \tag{25}$$

When $J_0 a \neq 0$,

$$\operatorname{str}_M \ o(a_{[-h_a]}b) x^{J_0} q^{L_0} = \sum_{n=1}^{+\infty} E_n \begin{bmatrix} e^{2\pi i h_a} \\ x^Q \end{bmatrix} \operatorname{str}_M o(a_{[-h_a+n]}b) x^{J_0} q^{L_0} \ .$$

Tools: flavored modular differential equations

- **Zhu's recursion formula**: for some nice null $\mathcal{N}$, the null state equation turns into a **FMLDE** for ch_M [Gaberdiel, Keller][Gaberdiel, Lang][Beem, Rastelli][Beem, Peelaers][YP, Wang, Zheng][Zheng, YP, Wang]

$$\mathcal{D}(D_q^{(k)}, D_{b_j})\,\mathrm{ch}_M = 0 \ , \tag{26}$$

where coefficients are **twisted Eisenstein**.

- Example: in $\beta\gamma$ system

$$\mathcal{N} = T - \frac{1}{2}\left((\beta\partial\gamma) - (\gamma\partial\beta)\right) \tag{27}$$

$$\Rightarrow \left(D_q^{(1)} - E_2\begin{bmatrix}-1\\ b\end{bmatrix}\right)\mathrm{ch} = 0 \ . \tag{28}$$

Tools: flavored modular differential equations

- For any associated VOA $\mathbb{V}(\mathcal{T})$: **at least one** such null $\mathcal{N}$ is expected to exist [Beem, Rastelli].
- Argument: stress tensor T of $\mathbb{V}(\mathcal{T})$ should be **nilpotent** up to a null state $\mathcal{N}_T$ and $\varphi \in C_2(\mathbb{V}(\mathcal{T}))$: $\exists k \in \mathbb{N}_{\geq 1}$

$$L_{-2}^{k}|0\rangle = \varphi + |\mathcal{N}_T\rangle \ , \qquad C_2(\mathbb{V}(\mathcal{T})) := \langle a_{-h[a]-1}b\rangle \ . \quad (29)$$

$\Rightarrow$ Special null state $\mathcal{N}_T$ associated with the **nilpotency** of T

- k value: hard to determine in general

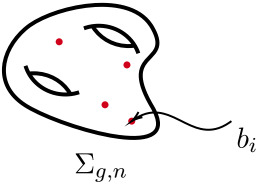
Tools: flavored modular differential equations

- All module characters of $\mathbb{V}(\mathcal{T})$ satisfy one FMLDE from $\mathcal{N}_T$ [Beem, Rastelli][Kaidi, et.al.].
- There may be **other** nulls $\Rightarrow$ other FMLDEs that kill all the module characters
- Idea: look for **simultaneous** exact solutions to all these flavored MDEs

Non-local operators, characters and FMLDEs

A_1 theory of class-$\mathcal{S}$

- A_1 theories of class-$\mathcal{S}$ $\mathcal{T}_{g,n} := \mathcal{T}[\Sigma_{g,n}]$



- Apply integration formula: **closed-form** Schur index for all g, n [YP, Peelaers][Bourdier, Drukker, Felix; Hatsuda, Okazaki for $\mathcal{N} = 4$]

$$\mathcal{I}_{g,n}(b) = \frac{i^n}{2}\frac{\eta(\tau)^{n+2g-2}}{\prod_{i=1}^{n}\vartheta_1(2\mathfrak{b}_i)}\sum_{k=1}^{n+2g-2}\lambda_k^{(n+2g-2)}\sum_{\alpha=\pm}\left(\prod_{i=1}^{n}\alpha_i\right)E_k\begin{bmatrix}(-1)^n\\ \prod_{i=1}^{n}b_i^{\alpha_i}\end{bmatrix}$$

$$\mathcal{I}_{g,n=0} = \frac{1}{2}\eta(\tau)^{2g-2}\sum_{k=1}^{g-1}\lambda_{2k}^{(2g-2)}\left(E_{2k} + \frac{B_{2k}}{(2k)!}\right)$$

λ's are **rational numbers** defined by simple recursions.

- Abbreviation

$$\mathcal{I}_{g,n} \propto \frac{\eta(\tau)^{2g-2+n}}{\prod_{j=1}^{n} \vartheta_1(2\mathfrak{b}_j)} \sum_{k=1}^{2g-2+n} \lambda_k^{(2g-2+n)} \mathbf{E}_k(b) \ , \tag{30}$$

where

$$\mathbf{E}_k(b) \coloneqq \sum_{\alpha_i=\pm} \Big(\prod_{i=1}^{n} \alpha_i\Big) E_k \left[\begin{matrix} (-1)^n \\ \prod_{j=1}^{n} b_j^{\alpha_j} \end{matrix}\right] \ . \tag{31}$$

- Manifest S-duality: permutation invariant between $\mathfrak{b}_i$

Vortex surface defect

- Focus on A_1 theories $\mathcal{T}_{g,n}$
- Vortex defect from Higgsing (with **vorticity k**) $\sim$ **poles** $b_i = q^{\frac{k+1}{2}}$ of $\mathcal{I}_{g,n+1}$ [Gaiotto, Rastelli, Razamat]
- Vortex defect index with vorticity k [Cordova, Gaiotto, Shao][Alday, et.al.][Nishinaka, Sasa, Zhu]

$$\mathcal{I}^{\text{vortex}}_{g,n}(k) = q^{-\frac{(k+1)^2}{2}} \operatorname*{Res}_{b \to q^{\frac{k+1}{2}}} \frac{2\eta(\tau)^2}{b} \mathcal{I}_{g,n+1}(b) \ . \tag{32}$$

Vortex surface defect

- **Closed-form** for all $\mathcal{I}^{\text{vortex}}_{g,n}(k)$ [YP, Peelaers]

$$\frac{\eta(\tau)^{n+2g-2}}{\prod_{i=1}^{n}\vartheta_1(2\mathfrak{b}_i)}\sum_{\ell=1}^{n+1+2g-2}\tilde{\lambda}_\ell^{(g,n)}(k)\sum_{\vec{\alpha}=\pm}\left(\prod_{i=1}^{n}\alpha_i\right)E_\ell\begin{bmatrix}(-1)^{n+k}\\ \prod_i b_i^{\alpha_i}\end{bmatrix},$$

- $k =$ even: solve all the known FMLDEs
- $k =$ odd: solve all the known twisted FMLDEs, thanks to some distinguished feature present in the closed-form

$$E_\ell\begin{bmatrix}(-1)^n\\ \prod_i b_i^{\alpha_i}\end{bmatrix}\xrightarrow{\text{odd } k}E_\ell\begin{bmatrix}-(-1)^n\\ \prod_i b_i^{\alpha_i}\end{bmatrix}\tag{33}$$

- In particular: for $\mathcal{I}_{g,0}$, there are solutions

$$\eta(\tau)^{2g-2},\eta(\tau)^{2g-2}E_2(\tau),\ldots,\eta(\tau)^{2g-2}E_{2g-2}\tag{34}$$

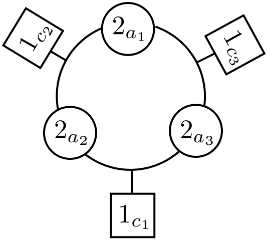
Gukov-Witten type defect

- Singular BPS background values for dynamical $\mathcal{N} = 2$ vector multiplets [Gukov, Witten]
- Schur index with Gukov-Witten surface defects: certain combinations of residues of the integrand

$$\mathcal{I} = \int \frac{d\mathbf{a}}{2\pi i \mathbf{a}} \mathcal{Z}(\mathbf{a}, \mathbf{b}), \qquad \mathcal{I}^{\mathsf{GW}} \sim \operatorname{Res} \mathcal{Z}(\mathfrak{a}, \mathbf{b}) \tag{35}$$

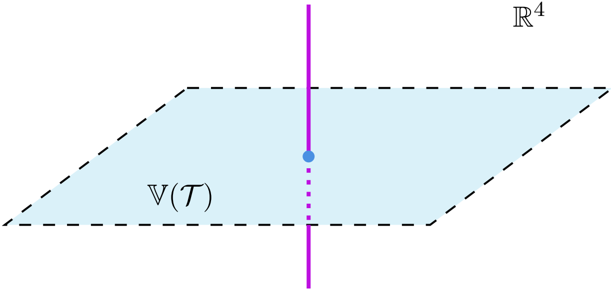
- Expect to be **module character**, and solve FMLDEs
- Checked for $(g, n) = (1, 0), (1, 1), (0, 4), (1, 2), (2, 0)$ [Zheng, YP, Wang]
- Proved for $\mathcal{N} = 4$ $SU(N)$ SYM [YP, Wang, Zheng]: based on a free field realization in [Bonetti, Meneghelli, Rastelli] where null state **vanish identically**

- The residue $\mathrm{Res}\,\mathcal{Z}(\mathfrak{a},\mathbf{b})$ also arise as $\mathcal{N}=(0,2)$ elliptic genus [Nawata, YP, Zheng] of circular $SU(2)$ quivers



Wilson line index/correlators

- $SU(2)$ half/full Wilson line from origin to infinity, orthogonal to $\mathbb{V}(\mathcal{T})$ plane [Cordova, Gaiotto, Shao]



- Schur index in the presence of Wilson line: schematic form

$$\langle W_{\mathcal{R}} \rangle = \oint \left[\frac{d\mathbf{a}}{2\pi i \mathbf{a}} \right] \chi_{\mathcal{R}}(\mathbf{a}) \mathcal{Z}(\mathbf{a}, \mathbf{b}) \ . \qquad (36)$$

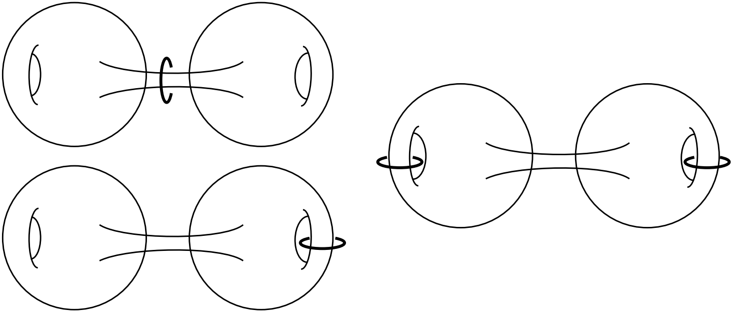
Wilson line index/correlators

- $\langle W \rangle$ counts **gauge-variant** local Schur operators $\mathcal{O}_W$: cancel the charge carried by the line
- Gauge invariant Schur operators $\mathcal{O}$ (those entering $\mathbb{V}(\mathcal{T})$) acts on the $\mathcal{O}_W$'s: form a **module** $\mathcal{W}$ of $\mathbb{V}(\mathcal{T})$ [Cordova, Gaiotto, Shao]

- Focus on class-$\mathcal{S}$ A_1 theories

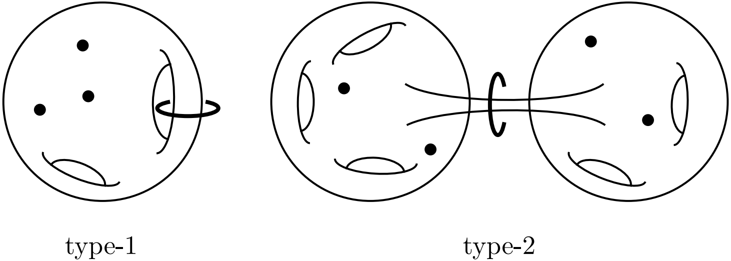
  See [Hatsuda, Okazaki] for results in $\mathcal{N}=4$ $SU(N)$ theories

- class-$\mathcal{S}$ A_1 theory $\mathcal{T}_{g,n}$: many choices of Wilson lines
  - different gauge-theory descriptions
  - many $SU(2)$ gauge groups for **each** gauge-theory description



Wilson line index/correlators

- Classification: **two types** of insertions of **one** $SU(2)$ Wilson lines



- type-1 only exits for $g \geq 1$
- Wilson line in spin-j representation: $\langle W_j \rangle^{(1)}_{g,n}$, $\langle W_j \rangle^{(2)}_{g_1,n_1;g_2,n_2}$
- Use integration formula to compute analytically

Wilson line index/correlators

- Type-1 index in $\mathcal{T}_{g,n}$:

$$\langle W_{j\in\mathbb{Z}}\rangle^{(1)}_{g,n} \tag{37}$$
$$= \mathcal{I}_{g,n} - \frac{1}{2}\prod_{i=1}^{n}\frac{i\eta(\tau)}{\vartheta_1(2\mathfrak{b}_i)}\sum_{\substack{m=-j\\m\neq 0}}^{j}\left[\frac{\eta(\tau)}{q^{\frac{m}{2}}-q^{-\frac{m}{2}}}\right]^{2g-2}\prod_{i=1}^{n}\frac{b_i^m-b_i^{-m}}{q^{\frac{m}{2}}-q^{-\frac{m}{2}}}\ .$$

$$\langle W_{j\in\mathbb{Z}+\frac{1}{2}}\rangle^{(1)}_{g,n} = 0\ . \tag{38}$$

- Among all choices of Wilson-line
  - **Independent** of the gauge-theory description
  - **Independent** of the specific chosen $SU(2)$ gauge group
- Linear combination of FMLDE solutions

$$\mathcal{I}_{g,n}, \quad \text{and} \quad \operatorname{Res}\mathcal{Z} = \prod_{i=1}^{n}\frac{i\eta(\tau)}{\vartheta_1(2\mathfrak{b}_i)}\ . \tag{39}$$

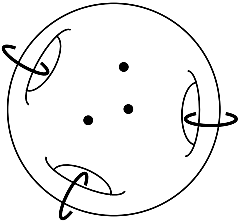
Wilson line index/correlators

- Type-1 correlator: still simple

$$\langle \prod_a W_{j\in\mathbb{Z}} \rangle^{(1)}_{g,n} \tag{40}$$

$$= \mathcal{I}_{g,n} - \frac{1}{2}\prod_{i=1}^{n} \frac{i\eta(\tau)}{\vartheta_1(2\mathfrak{b}_i)} \sum_{\substack{m=-\max(j)\\ m\neq 0}}^{\max(j_a)} \left[\frac{\eta(\tau)}{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}\right]^{2g-2} \prod_{i=1}^{n} \frac{b_i^m - b_i^{-m}}{q^{\frac{m}{2}} - q^{-\frac{m}{2}}} \ .$$

$$\langle \prod_a W_{j_a} \rangle^{(1)}_{g,n} = 0 \text{ if any one } j_a \in \mathbb{Z} + \frac{1}{2} \ . \tag{41}$$



Wilson line index/correlators

- Type-2 in $\mathcal{T}_{1,2}$

$$\begin{aligned}
&\langle W_j\rangle^{(2)}_{1,1;0,3} \qquad (42)\\
&= \delta_{j\in\mathbb{Z}}\mathcal{I}_{1,2} - \frac{\eta(\tau)^2}{\prod_{i=1}^{2}\vartheta_1(2\mathfrak{b}_i)}\sum_{\substack{m=j\\ m\neq 0}}^{+j}(q^m+q^{-m})\prod_{i=1,2}\frac{b_i^{2m}-b_i^{-2m}}{q^m-q^{-m}}\\
&\quad - \frac{\eta(\tau)^2}{2\prod_{i=1}^{2}\vartheta_1(2\mathfrak{b}_i)}\sum_{\alpha=\pm}\left(\alpha E_1\begin{bmatrix}1\\ b_1 b_2^{\alpha}\end{bmatrix}\sum_{\substack{m=-j\\ m\neq 0}}^{+j}\frac{(b_1b_2^{\alpha})^{2m}-(b_1b_2^{\alpha})^{-2m}}{q^m-q^{-m}}\right).
\end{aligned}$$

Linear combination of FMLDEs solutions

$$\mathcal{I}_{1,2}, \quad \mathrm{Res}\,\mathcal{Z} = \frac{\eta(\tau)^2}{\prod_{i=1}^{2}\vartheta_1(2\mathfrak{b}_i)}, \quad (\mathrm{Res}\,\mathcal{Z})E_1\begin{bmatrix}1\\ b_1b_2^{\pm}\end{bmatrix}.$$

Wilson line index/correlators

- Type-2 index in all $\mathcal{T}_{g=g_1+g_2,0}$

$$\langle W_j \rangle^{(2)}_{g_1,1;g_2,1} = \sum_{k=0}^{g-1} R_k(q) \eta(\tau)^{2g-2} E_{2k} \ . \tag{43}$$

which $\eta(\tau)^{2g-2} E_{2k}$ are all MDE solutions w.r.t. $\mathcal{T}_{g,0}$.

Wilson line index/correlators

- General type-2 index: closed-form, but not-so elegant

$$\langle W_{j\in\mathbb{Z}}\rangle^{(2)}_{g_1,n_1;g_2,n_2} = \mathcal{I}_{g_1+g_2,n_1+n_2-2} + \sum_{\substack{m=-j \\ m\neq 0}}^{j} R(\mathbf{b},q)\,\mathrm{Res}\,\mathcal{Z}E_k\cdots\ .$$

- $\langle W \rangle$ are **not** solutions to the FMLDEs: $\mathcal{W}$ is **not** a module of the **simple** $\mathbb{V}(\mathcal{T})$
- Observation: $\langle W \rangle$ **contain** solutions to the FMLDEs: removing rational coefficients

$$\operatorname{Res} \mathcal{Z}(\mathbf{a}, \mathbf{b}) E_{k_1} \begin{bmatrix} \pm 1 \\ \mathbf{b} \end{bmatrix} E_{k_2} \begin{bmatrix} \pm 1 \\ \mathbf{b} \end{bmatrix} \cdots \tag{44}$$

- New solutions **not** from surface defect index, e.g., $(g, n) = (1, 2)$

$$\frac{\eta(\tau)}{\prod_{i=1}^{2} \vartheta_1(2\mathfrak{b}_i)} E_1 \begin{bmatrix} +1 \\ b_1 b_2^{\pm 1} \end{bmatrix} . \tag{45}$$

Modular property

Modular transformation

- Rational CFT unflavored characters ch_i:

$$\mathrm{ch}_i(-\frac{1}{\tau}) = \sum_i S_{ij}\,\mathrm{ch}_j(\tau)\ . \tag{46}$$

- Chiral algebra $\mathbb{V}(\mathcal{T})$ in 4d/2d correspondence
  - ch_0: Schur index
  - ch_i: (believed to be) defect index
  - non-unitary, often non-rational, logarithmic: the full set of ch_i **unknown**, which ch_i participate **unknown**
- Exploration: making use of closed-form formula

Example: $\mathcal{N}=4$ $SU(2)$

- 2d $\mathcal{N}=4$ small superconformal algebra $\supset \widehat{\mathfrak{su}}(2)_{k=-3/2}$
- The (extended) Schur index

$$\mathcal{I} = y^{k=-3/2}\frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}E_1\begin{bmatrix}-1\\ b\end{bmatrix} . \tag{47}$$

- modular S-transformation

$$\mathfrak{y}\to\mathfrak{y}-\frac{\mathfrak{b}^2}{\tau}, \qquad \tau\to-\frac{1}{\tau}, \qquad \mathfrak{b}\to\frac{\mathfrak{b}}{\tau} . \tag{48}$$

- STS-transformation,

$$STS\mathcal{I} = -\tau\mathcal{I} - \mathfrak{b}_1 i y^k\frac{\vartheta_4(\mathfrak{b}_1)}{\vartheta_1(2\mathfrak{b}_1)} = -\tau\mathcal{I} - \mathfrak{b}_1\Delta\mathcal{I} \tag{49}$$

$$\Delta\mathcal{I} := (b^2q)^k\mathcal{I}(\mathfrak{b}+\tau)+\mathcal{I}(\mathfrak{b}) .$$

Example: $SU(2)$ SQCD

- $\widehat{\mathfrak{so}}(8)_{-2} \supset \widehat{\mathfrak{su}}(2)_{-2} \oplus \widehat{\mathfrak{su}}(2)_{-2} \oplus \widehat{\mathfrak{su}}(2)_{-2} \oplus \widehat{\mathfrak{su}}(2)_{-2}$
- The index

$$\mathcal{I}_{0,4} = \prod_{i=1}^{4} y_i^{-2} \sum_{j=1}^{4} E_1 \begin{bmatrix} -1 \\ m_j \end{bmatrix} \frac{i\vartheta_1(2\mathfrak{m}_j)}{\eta(\tau)} \prod_{\ell \neq j} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j + \mathfrak{m}_\ell)} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j - \mathfrak{m}_\ell)} ,$$

where the flavor fugacities

$$m_1 = b_1 b_2, \quad m_2 = \frac{b_1}{b_2}, \quad m_3 = b_3 b_4, \quad m_4 = \frac{b_3}{b_4} \ . \tag{50}$$

- S-transformation

$$\mathfrak{y}_i \to \mathfrak{y}_i - \frac{\mathfrak{b}_i^2}{\tau}, \quad \mathfrak{b}_i \to \frac{\mathfrak{b}_i}{\tau}, \quad \tau \to -\frac{1}{\tau} \ . \tag{51}$$

- S-transformation

$$S\mathcal{I}_{0,4} = i\tau \mathcal{I}_{0,4} + 2i \sum_{j=1}^{4} \mathfrak{m}_j R_j \ , \tag{52}$$

where R_j are residues of the integrand,

$$R_j \equiv \underset{a \to m_j^{\pm 1} q^{\frac{1}{2}}}{\text{Res}} (\text{integrand})$$
$$= \frac{i}{2} \prod_{i=1}^{4} y_i^{-2} \frac{\vartheta_1(2\mathfrak{m}_j)}{\eta(\tau)} \prod_{\ell \neq j} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j + \mathfrak{m}_\ell)} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j - \mathfrak{m}_\ell)} \ . \tag{53}$$

Example: $SU(2)$ SQCD

- In terms of difference operators,

$$S\mathcal{I}_{0,4} = i\tau\mathcal{I}_{0,4} + \sum_{j=1}^{4}(-i\mathfrak{b}_j)\Delta_j\mathcal{I}_{0,4}\ , \tag{54}$$

where $\Delta_j\mathcal{I} = (b_j^2 q)^{-2}\mathcal{I}(\mathfrak{b}_j + \tau) - \mathcal{I}$.

Other examples

- $g = 1, n = 2$

$$S\mathcal{I}_{1,2} = -\tau^2 \mathcal{I}_{1,2} + \tau \sum_{i=1}^{2} \mathfrak{b}_i \Delta_i \mathcal{I}_{1,2} - \mathfrak{b}_1 \mathfrak{b}_2 \Delta_1 \Delta_2 \mathcal{I}_{1,2} \ . \tag{55}$$

- $g = 1, n = 4$

$$\begin{aligned} S\mathcal{I}_{1,4} = \ & \tau^4 \left(\mathcal{I}_{1,4} + \frac{1}{6}\eta(\tau)^2 \mathcal{I}_{0,4}\right) - \tau^3 \sum_{j=1}^{4} \mathfrak{b}_j \left(\Delta_j \mathcal{I}_{1,4} - \frac{1}{2}\eta(\tau)^2 \mathcal{I}_{0,4}\right) \\ & + \tau^2 \sum_{i,j} \left[\frac{1}{2} \mathfrak{b}_i \mathfrak{b}_j \left(\Delta_i \Delta_j \mathcal{I}_{1,4} - \Delta_i \Delta_i \Delta_j \mathcal{I}_{1,4}\right) - \frac{1}{6}\eta(\tau)^2 \mathcal{I}_{0,4}\right] \\ & - \tau \left(\frac{1}{6} \sum_{i,j,k} \mathfrak{b}_i \mathfrak{b}_j \mathfrak{b}_k \Delta_i \Delta_j \Delta_k \mathcal{I}_{g,n} + \frac{1}{6} \sum_{i=1}^{4} \mathfrak{b}_i \Delta_i \Delta_i \Delta_i \mathcal{I}_{g,n}\right) \\ & + \mathfrak{b}_1 \mathfrak{b}_2 \mathfrak{b}_3 \mathfrak{b}_4 \Delta_1 \Delta_2 \Delta_3 \Delta_4 \mathcal{I}_{1,4} \ . \end{aligned}$$

Other examples

- $g = 2, n = 2$

$$
\begin{aligned}
& S\mathcal{I}_{2,2} \qquad (56) \\
&= i\tau^5\left(\mathcal{I}_{g,n} + \frac{i}{6}\eta(\tau)^2\mathcal{I}_{g-1,n}\right) - i\tau^4 \sum_{j=1}^{n} \mathfrak{b}_j \left(\Delta_j \mathcal{I}_{g,n} - \frac{i}{2}\eta(\tau)^2\mathcal{I}_{g-1,n}\right) \\
&\quad + \frac{i\tau^3}{2} \sum_{j,k} \mathfrak{b}_j \mathfrak{b}_k (\Delta_j \Delta_k \mathcal{I}_{g,n} - \Delta_j \Delta_j \Delta_k \mathcal{I}_{g,n}) - \frac{i\tau^3}{6}\eta(\tau)^2\mathcal{I}_{g-1,n} \\
&\quad - \frac{i\tau^2}{6} \sum_{j,k,\ell} \mathfrak{b}_j \mathfrak{b}_k \mathfrak{b}_\ell \Delta_j \Delta_k \Delta_\ell \mathcal{I}_{g,n} + \frac{i\tau^2}{6} \sum_{j} \mathfrak{b}_j \Delta_j \Delta_j \Delta_j \mathcal{I}_{g,n} \\
&\quad + \frac{\eta(\tau)^4}{\prod_{j=1}^{2} \vartheta_1(2\mathfrak{b}_j)} \frac{i\tau \mathfrak{b}_1 \mathfrak{b}_2 (-1 + 5\tau^2 + \sum_{j=1}^{2}(3\tau \mathfrak{b}_j + \mathfrak{b}_j^2))}{3} \, .
\end{aligned}
$$

- For general $\mathcal{I}_{g,n}$: $S\mathcal{I}_{g,n}$ (or $STS\mathcal{I}_{g,n}$) are combinations of $\mathcal{I}$ and $\Delta_j \Delta_k \ldots$ acting on $\mathcal{I}_{g,n}, \mathcal{I}_{g-1,n}, \mathcal{I}_{g-2,n}, \ldots$

Modular transformation and vortex defect

- The modular orbit of $\mathcal{I}_{g,n}$ spans a linear space $\mathcal{V}_{g,n}$
- **Claim**: vortex defect index $\mathcal{I}^{\text{vortex}}_{g,n}(k = \textbf{even}) \in \mathcal{V}_{g,n}$,

$$\mathcal{I}^{\text{defect}}_{g,n}(k = \text{even}) \propto \frac{\eta(\tau)^{2g-2+n}}{\prod_{j=1}^{n} \vartheta_1(2\mathfrak{b}_j)} \sum_{\ell=1}^{2g-2+n+1} \tilde{\lambda}_{\ell}^{2g-2+n+1}(k+1)\mathbf{E}_{\ell}(b) \ , \quad (57)$$

where

$$\mathbf{E}_{\ell}(b) := \sum_{\alpha_i = \pm} \left(\prod_{j=1}^{n} \alpha_j \right) E_{\ell} \begin{bmatrix} (-1)^n \\ \prod_{j=1}^{n} b_j^{\alpha_j} \end{bmatrix} , \quad (58)$$

the **same** expression in the original index $\mathcal{I}_{g,n}$

Modular transformation and vortex defect

- Proof:

$$\mathbf{T}^{\ell-1} S\mathbf{E}_\ell(b) \xrightarrow{S,STS} \ell!\mathbf{E}_\ell(b) \ , \tag{59}$$

which leads to crucial relations, e.g., for $n =$ even,

$$(\mathbf{T}_{g,n})^{3g-3+n} S\mathcal{I}_{g,n} = (3g-3+n)! i^{3g-3+n} \mathbf{E}_{2g-2+n}(b) \ . \tag{60}$$

where

$$\mathbf{T}_{g,n} f \coloneqq e^{-\frac{\pi i (g-n-1)}{6}} Tf - f \ . \tag{61}$$

Modular transformation and vortex defect

- Example: $g = 1, n = 4$,

$$\mathcal{I}^{\mathsf{defect}}_{g,n}(\kappa = 2) = -3\mathcal{I}_{g,n} + \frac{1}{4}(-1)^{2/3}(\mathbf{T}_{g,n})^4 S\mathcal{I}_{g,n} \ . \qquad (62)$$

- Example: $g = 2, n = 2$,

$$\mathcal{I}^{\mathsf{defect}}_{g,n}(\kappa = 2) = 3\mathcal{I}_{g,n} + \frac{i}{20}(\mathbf{T}_{g,n})^5 S\mathcal{I}_{g,n} \ . \qquad (63)$$

- Example: $g = 3, n = 2$,

$$\begin{aligned}\mathcal{I}^{\mathsf{defect}}_{g,n}(\kappa = 2) = 2\mathcal{I}_{g,n} - \frac{1}{8064}(\mathbf{T}_{g,n})^{3g-3+n}\mathcal{I}_{g,n} \qquad (64)\\ - \frac{1}{720}(\mathbf{T}_{g,n})^{3g-3+n-2} S(\mathcal{I}_{g,n} - \frac{1}{8!}(\mathbf{T}_{g,n})^{3g-3+n}\mathcal{I}_{g,n}) \ .\end{aligned}$$

Modular transformation and line defect

- In $S\mathcal{I}_{g,n}$: Δ_i action on $\mathcal{I}_{g,n}$
- Combinations of $\Delta_i \ldots \mathcal{I}_{g,n}$: related to line defect index
- Example: type-1 correlator

$$
\begin{aligned}
&\langle \prod_a W_{j\in\mathbb{Z}} \rangle^{(1)}_{g,n} \qquad (65) \\
&= \mathcal{I}_{g,n} - \frac{1}{2} \prod_{i=1}^{n} \frac{i\eta(\tau)}{\vartheta_1(2\mathfrak{b}_i)} \sum_{\substack{m=-\max(j) \\ m\neq 0}}^{\max(j_a)} \left[\frac{\eta(\tau)}{q^{\frac{m}{2}} - q^{-\frac{m}{2}}} \right]^{2g-2} \prod_{i=1}^{n} \frac{b_i^m - b_i^{-m}}{q^{\frac{m}{2}} - q^{-\frac{m}{2}}} \ .
\end{aligned}
$$

- Factor in the second term

$$
\prod_{j=1}^{n} \frac{i\eta(\tau)}{\vartheta_1(2\mathfrak{b}_j)} = (\Delta_1)^{2g-2} \Delta_1 \Delta_2 \ldots \Delta_n \mathcal{I}_{g,n} \ . \qquad (66)
$$

Modular transformation and line defect

- Example: type-2 index in $\mathcal{T}_{g=1,n=2}$

$$\begin{aligned}
&\langle W_j\rangle^{(2)}_{1,1;0,3} \qquad (67)\\
&= \delta_{j\in\mathbb{Z}}\mathcal{I}_{1,2} - \frac{\eta(\tau)^2}{\prod_{i=1}^{2}\vartheta_1(2\mathfrak{b}_i)} \sum_{\substack{m=j\\ m\neq 0}}^{+j}(q^m+q^{-m})\prod_{i=1,2}\frac{b_i^{2m}-b_i^{-2m}}{q^m-q^{-m}}\\
&\quad - \frac{\eta(\tau)^2}{2\prod_{i=1,2}\vartheta_1(2\mathfrak{b}_i)}\sum_{\alpha=\pm}\left(\alpha E_1\begin{bmatrix}1\\ b_1 b_2^\alpha\end{bmatrix}\sum_{\substack{m=-j\\ m\neq 0}}^{+j}\frac{(b_1b_2^\alpha)^{2m}-(b_1b_2^\alpha)^{-2m}}{q^m-q^{-m}}\right).
\end{aligned}$$

where

$$\frac{\eta(\tau)^2}{\prod_{j=1}^{2}\vartheta_1(2\mathfrak{b}_j)} = \Delta_1\Delta_2\mathcal{I}_{g,n} \qquad (68)$$

$$\frac{\eta(\tau)^2}{\prod_{j=1}^{2}\vartheta_1(2\mathfrak{b}_j)}\alpha E_1\begin{bmatrix}1\\ b_1b_2^\alpha\end{bmatrix} = \frac{1}{2}(\Delta_1+\alpha\Delta_2)\mathcal{I}_{g,n}\ . \qquad (69)$$

- Example: type-2 index in $\mathcal{T}_{2,2}$

$$\begin{aligned}\langle W\rangle^{(2)}_{1,2;1,2} =& -\frac{1}{2}\frac{\eta(\tau)^4}{\prod_{j=1}^{2}\vartheta_1(2\mathfrak{b}_j)} \\ &\times\Bigg[\sum_{\alpha=\pm}\frac{q^4(1+q^4)(b_1^4+\alpha b_2^4)(-\alpha+b_1^4b_2^4)}{(-1+q^4)^3b_1^4b_2^4}E_1\begin{bmatrix}1\\ b_1b_2^\alpha\end{bmatrix} \\ &+\frac{q^4(b_1^4+b_2^4)(1+b_1^4b_2^4)}{(-1+q^4)^2b_1^4b_2^4}\left(E_2\begin{bmatrix}1\\ b_1b_2\end{bmatrix}-E_2\begin{bmatrix}1\\ b_1/b_2\end{bmatrix}\right)\Bigg]\ .\end{aligned} \tag{70}$$

- Factors can be rewritten in **difference operators**,

$$\begin{aligned}\frac{\eta(\tau)^4}{\prod_{j=1}^{2}\vartheta_1(2\mathfrak{b}_j)}E_1\begin{bmatrix}1\\ b_1/b_2\end{bmatrix} =& -\frac{1}{2}(\Delta_1\Delta_1\Delta_2-\Delta_1\Delta_2\Delta_2)\mathcal{I}_{2,2} \\ \frac{\eta(\tau)^4}{\prod_{j=1}^{2}\vartheta_1(2\mathfrak{b}_j)}\sum_{\alpha}\alpha E_2\begin{bmatrix}1\\ b_1b_2^\alpha\end{bmatrix} =& [(\Delta_1)^3-(\Delta_1)^2]\mathcal{I}_{2,2}\ .\end{aligned} \tag{71}$$

High temperature behavior

- Rewrite everything in terms of the modular-transformed variables $\tilde{\mathfrak{b}} = \frac{\mathfrak{b}}{\tau}$, $\tilde{\tau} = -\frac{1}{\tau}$,

$$\eta(\tau) = \sqrt{-i\tilde{\tau}}\eta(\tilde{\tau}), \quad \vartheta_1(\mathfrak{b}|\tau) = \sqrt{-i\tilde{\tau}}e^{\frac{\pi i \tilde{\mathfrak{b}}^2}{\tilde{\tau}}}\vartheta_1(\tilde{\mathfrak{b}}|\tilde{\tau}) \ , \tag{72}$$

$$E_n \begin{bmatrix} 1 \\ b \end{bmatrix} (\tau) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!} \tilde{\mathfrak{b}}^k \tilde{\tau}^{n-k} E_{n-k} \begin{bmatrix} 1 \\ \tilde{b} \end{bmatrix} (\tilde{\tau}) \ , \tag{73}$$

$$\vdots \tag{74}$$

High temperature behavior

- $\mathcal{I}_{g,n=\text{even}}$ as series expansion in $\tilde{q}$, and $\tilde{\tau}, \tilde{\mathfrak{b}}_j$,

$$\mathcal{I}_{g,n} = \frac{1}{2}(-i\tilde{\tau})^{g-1}\frac{\eta(\tilde{\tau})^{2g-2+n}}{\prod_{j=1}^{n}\vartheta_1(2\tilde{\mathfrak{b}}_j|\tilde{\tau})}\sum_{k=1}^{2g-2+n}\lambda_k^{(2g-2+n)}\Bigg(\sum_{\ell=0}^{k}\frac{1}{\ell!}\sum_{\alpha_j=\pm}\Big(\prod_{j=1}^{n}\alpha_j\Big)\Big(\sum_{j=1}^{n}\alpha_j\tilde{\mathfrak{b}}_j\Big)^{\ell}(-\tilde{\tau})^{k-\ell}E_{k-\ell}\left[\frac{1}{\prod_j \tilde{b}^{\alpha_j}}\right]\Bigg)\,. \tag{75}$$

- The leading $\tilde{q}$ power (Cardy behavior)

$$\tilde{q}^{\frac{1}{12}(g-n-1)} = e^{-\frac{2\pi i}{\tau}\frac{g-n-1}{12}} = \tilde{q}^{-2(c_{\text{4d}}-a_{\text{4d}})} = \tilde{q}^{-\frac{1}{24}c_{\text{eff}}} = \tilde{q}^{-\frac{1}{24}(c_{\text{2d}}-24h_{\text{min}})}\ , \tag{76}$$

where

$$c_{\text{4d}} = \frac{1}{6}(5n+13g-13) = \frac{c_{\text{2d}}}{-12}, \quad a_{\text{4d}} = \frac{1}{24}(19n+53g-53)\ .$$

High temperature behavior

- Example: $\mathcal{N} = 4\ SU(2)$

$$e^{+\frac{3\pi i \tilde{\mathfrak{b}}^2}{\tilde{\tau}}}\mathcal{I}(\mathfrak{b}) = -\frac{\tilde{\mathfrak{b}}\vartheta_2(\tilde{\mathfrak{b}}|\tilde{\tau})}{\vartheta_1(2\tilde{\mathfrak{b}}|\tilde{\tau})} + \frac{i\tilde{\tau}}{2\pi}\frac{\vartheta_2'(\tilde{\mathfrak{b}}|\tilde{\tau})}{\vartheta_1(2\tilde{\mathfrak{b}}|\tilde{\tau})} \quad (77)$$

$$\xrightarrow{\tilde{\mathfrak{b}}\to 0} -\frac{1}{2\pi} - \frac{2\tilde{q}}{\pi} - \frac{6\tilde{q}^2}{\pi} + \ldots - \frac{\ln\tilde{q}}{8\pi} - \frac{3\tilde{q}\ln\tilde{q}}{2\pi} - \frac{9\tilde{q}^2\ln\tilde{q}}{2\pi} + \ldots\ .$$

agrees and improves the result in [Ardehali, Martone, Rossello]

High temperature behavior

- Example: $SU(2)$ SQCD

$$\mathcal{I}_{0,4} = \frac{1}{2}\frac{1}{-i\tilde{\tau}}\frac{\eta(\tau)^2}{\prod_{j=1}^{4}\vartheta_1(2\tilde{\mathfrak{b}}_j)}$$
$$\times \sum_{\alpha_j=\pm}\left(\prod_j \alpha_j\right)\left[-\frac{1}{2}(\alpha\cdot\tilde{\mathfrak{b}})^2 - \tilde{\tau}(\alpha\cdot\tilde{\mathfrak{b}})E_1\begin{bmatrix}1\\ \tilde{b}^{\alpha}\end{bmatrix}(\tilde{\tau})\right.$$
$$\left. + \tilde{\tau}^2 E_2\begin{bmatrix}1\\ \tilde{b}^{\alpha}\end{bmatrix}(\tilde{\tau})\right] .$$

Expanding as a $\tilde{q}$ series, agrees with [Ardehali, Martone, Rossello]

$$= \frac{\tilde{q}^{-\frac{5}{12}}}{120\pi}(1 + 250\tilde{q} + 4625\tilde{q}^2 + 44250\tilde{q}^3 + 305750\tilde{q}^4 + \ldots) \quad (78)$$
$$+60\ln\tilde{q}\mathcal{I}_{0,4} \ . \quad (79)$$

3d reduction

- 3d limit when $c_{\text{eff}} \sim c_{\text{4d}} - a_{\text{4d}} > 0$: $\tau \to +0i$ picks out the leading $\tilde{q}$ independent term
- Example: $g = 0, n = 4$, $j = 1, 2, 3, 4$

$$\mathcal{I}_{0,4} \to -\frac{\tilde{b}_1\tilde{b}_2\tilde{b}_3\tilde{b}_4}{\prod_j \left(1 - b_j^2\right)} \sum_{\alpha_j = \pm} \Big(\prod_{j=1}^{n} \alpha_j\Big)(\alpha \cdot \tilde{\mathfrak{b}}) \left(-\frac{1}{2} + \frac{1}{\prod_j (1 - \tilde{b}_j^{\alpha_j})}\right)$$

agrees with known result [Gadde, Yan][Benvenuti, Pasquetti],

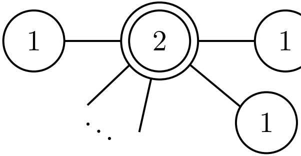
$$Z^{S^3} = \sum_{i=1}^{4} \frac{\mathfrak{m}_i 2\sinh(2\pi\mathfrak{m}_i)}{\prod_{j\neq i}(2\sinh\pi\mathfrak{m}_i)^2 - (2\sinh\pi\mathfrak{m}_j)^2} \ . \qquad (80)$$

3d reduction

- Example: $g = 0, n =$ even

$$\mathcal{I}_{0,n} \tag{81}$$
$$\rightarrow -\frac{\prod_{j=1}^{n} \tilde{b}_j}{\prod_{j=1}^{n}(1-\tilde{b}_j^2)} \sum_{k=1}^{2g-2+n} \sum_{\alpha_j=\pm} \lambda_k^{(2g-2+n)} \Big(\prod_{j=1}^{n} \alpha_j\Big) (\alpha \cdot \tilde{\mathfrak{b}})^{k-1} \times \left(-\frac{1}{2} + \frac{1}{1-\tilde{b}_j^{\alpha_j}}\right)$$

expected to equal Z^{S^3} of the 3d-mirror of the $SU(2) \times U(1)^n$ star-shaped quiver



Modular linear differential equations

MLDEs

- Special null $|\mathcal{N}_T\rangle$ exists in any associated VOA $\mathbb{V}(\mathcal{T})$ [Beem, Rastelli]
- **Additional null** may also exist
- Multiple FMLDEs (and their **unflavored** limit) govern the associated VOA

- Unflavored MLDEs

$$D_q^{(n)}\,\mathrm{ch}_i(\tau) + \sum_{r=0}^{n-1} \phi_{2r}(\tau) D_q^{(n-r)}\,\mathrm{ch}_i(\tau) = 0\ . \qquad (82)$$

- **Modularity**: **covariant** under suitable modular group, solution $\xrightarrow{\text{modular trans}}$ another solution
- **Question**: can ch_0 generate **all** unflavored solutions through modular transformation?
  Vortex defect index with even vorticity: already inside the modular orbit of ch_0

Unflavored MLDEs

- The flavored $\mathcal{I}_{g,n=\text{even},>0}$ exact formula: ingredients

$$\frac{\eta(\tau)^{2g-2+n}}{\prod_{j=1}^{n}\vartheta_1(2\mathfrak{b}_j)}, \quad \mathbf{E}_2(b), \quad \mathbf{E}_4(b), \quad ,\cdots \quad \mathbf{E}_{2g-2+n}(b)\ . \quad (83)$$

- They generate a linear space of dimension $\mathcal{V}_{g,n=\text{even},>0}$

$$\dim\mathcal{V}_{g,n} = \max(g,1)\sum_{k=2}^{2g-2+n}(k+1)\ . \quad (84)$$

- Suitably unflavoring **reduces** $\dim\mathcal{V}_{g,n}$ by

$$\delta_{g,n} = \begin{cases} \frac{2g-2+n}{2} & g=0 \\ (g-1)\left(\frac{2g-2+n}{2}\right)\left(\frac{2g-2+n}{2}+1\right) & g>0 \end{cases}\ . \quad (85)$$

Unflavored MLDEs

- The **unflavored** modular orbit has dimension

$$\dim \mathcal{V}^0_{g,n=\text{even},>0} = \dim \mathcal{V}_{g,n} - \delta_{g,n} \ , \tag{86}$$

agrees with [Beem, Sinh, Razamat] when $n =$ even and positive

$$(1-\delta_{g,0})g(2g+n-1) + \lfloor \frac{n-1}{2} \rfloor \left(g+n-1-\lfloor \frac{n-1}{2} \rfloor \right) . \tag{87}$$

- Observation: three quantities coincide [Beem, Rastelli][Beem, Sinh, Rastelli]
  - $\dim \mathcal{V}^{\text{unflavored}}_{g,n}$
  - minimal order ord of the unflavored MLDE possibly with **non-zero** Wronskian index
  - number of **rational** indicial roots of the minimal unflavored MDE with **zero** Wronskian index of order ord_T from $|\mathcal{N}_T\rangle$

Unflavored MLDEs

(g,n)	ord_T	$\dim \mathcal{V}^0_{g,n}$	ord	roots
$(0,4)$	2	2	2	$-\frac{5}{12}, \frac{7}{12}$
$(1,4)$	10	8	8	$\left(-\frac{1}{3}\right)_2, \frac{5}{3}, \left(\frac{2}{3}\right)_5$
$(1,6)$	21	15	15	$\left(-\frac{1}{2}\right)_2, \frac{5}{2}, \left(\frac{3}{2}\right)_5, \left(\frac{1}{2}\right)_7$
$(2,2)$	12	10	10	$\left(-\frac{1}{12}\right)_4, \left(\frac{11}{12}\right)_5, \frac{23}{12}$

- Example: $(g=1, n=4)$

$$\begin{aligned} & E_4 D_q^{(8)} - 14E_6 D_q^{(7)} - 720E_4^2 D_q^{(6)} - 21000E_4E_6 D_q^{(5)} \\ & + (294000E_6^2 - 351600E_4^3) D_q^{(4)} - 2788800E_4^2E_6 D_q^{(3)} \\ & + 2000(10120E_4^4 + 3087E_4E_6^2) D_q^{(2)} \\ & + 28000(43814E_4^3E_6 - 3087E_6^3) D_q^{(1)} \\ & + 2048000E_4^2(1860E_4^3 + 3283E_6^2) \end{aligned}$$

- $n=0$, or $n=$ odd: similar separate analysis

Flavored MLDEs: quasi-modularity

- Null states in $\mathbb{V}(\mathcal{T}) \Rightarrow$ Flavored MLDEs
- FMLDEs
  - hard to write down
  - non-trivial modularity: quasi-modularity of $E_k\left[\begin{smallmatrix}\pm 1\\ b\end{smallmatrix}\right]$
- Observations: FMLDEs **mix** under modular transformation

$$\text{weight-}k \xrightarrow{S} \tau^k\text{weight-}k \oplus \mathfrak{b}\tau^{k-1}\text{weight-}(k-1) \oplus \mathfrak{b}\mathfrak{b}\tau^{k-2}\text{weight-}(k-2) \oplus \ldots \ . \tag{88}$$

- Alternative interpretation: hidden modular structure among null states

$$|\mathcal{N}_k\rangle \xrightarrow{S} |\mathcal{N}_k\rangle \oplus |\mathcal{N}_{k-1}\rangle \oplus |\mathcal{N}_{k-2}\rangle \oplus \ldots \ . \tag{89}$$

FMLDEs and quasi-modularity

- Admissible Kac-Moody $\widehat{\mathfrak{su}}(2)_{-4/3}$
- $|\mathcal{N}_3^+\rangle$: affine null state, weight-3

$$|\mathcal{N}_3^+\rangle = \left(-\frac{13}{9}J^+_{-3} + \frac{11}{3}J^3_{-2}J^+_{-1}\cdots\right)|0\rangle \qquad (90)$$

- $|\mathcal{N}_T\rangle$: weight-4, an **affine descendant** of $\mathcal{N}_3^+$,

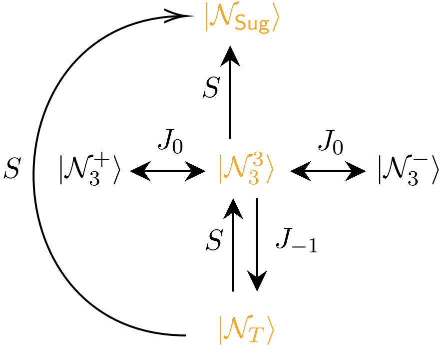
$$|\mathcal{N}_T\rangle = \frac{9}{8}\Big(2J^3_{-1}|\mathcal{N}_3^3\rangle + J^-_{-1}|\mathcal{N}_3^+\rangle + J^+_{-1}|\mathcal{N}_3^-\rangle\Big)\ . \qquad (91)$$

- The Sugawara condition

$$|\mathcal{N}_{\mathsf{Sug}}\rangle = L_{-2}|0\rangle - \frac{1}{2(k+h^\vee)}\sum_{A,B}K_{AB}J^A_{-1}J^B_{-1}|0\rangle\ . \qquad (92)$$

FMLDEs and Modularity

- Modular structure among nulls



- **Quasi-modularity** strongly constrains the chiral algebra
- Affine Kac-Moody $\widehat{\mathfrak{g}}_k$ with **weight-four quasi-modularity**: fixes root structure of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{f}_2, \mathfrak{d}_4, \mathfrak{g}_2, \mathfrak{e}_{6,7,8} \ , \qquad k = 1, -\frac{h^\vee}{6} - 1 \ . \tag{93}$$

- $k = -\frac{h^\vee}{6} - 1$: all characters are determined by the nulls in the modular orbit
- $k = 1$: all characters are determined by the nulls in the modular orbit and the Sugawara condition

FMLDEs and shift

- All FMLDEs are quasi-invariant under shifts/spectral flow

$$\mathfrak{b}_i \to \mathfrak{b}_i + n_i \tau, \qquad \mathfrak{y} \to \mathfrak{y} + \sum_i K^{ij} n_j \mathfrak{b}_i + \frac{1}{2} K^{ij} n_i n_j \tau \ .$$

- Conjecture:

vacuum characters $\to$ all highest weight characters

Outlook

- Significance of $|\mathcal{N}_T\rangle$: how much does it determine other null states and $\mathbb{V}$?
- Physical origin of unflavored MLDEs with **non-zero** Wronskian index, e.g., for $(E_8)_1$

$$D_q^{(1)}\,\mathrm{ch} - \left(\frac{D_q^{(1)}\,\mathrm{ch}}{\mathrm{ch}}\right)\mathrm{ch} = 0\ . \tag{94}$$

  Their **flavored** version?
- Constraints from higher weight quasi-modularity
- Verlinde formula?

Thank you