Peierls bracket and gravitational gressing in Jackiw-Teitelboim gravity

Jie-qiang Wu (MIT)

Based on in-going work with Daniel Harlow
also see arXiv: 1906.08616

Fudan

June 2nd, 2020

- In this work, we revisit the **Peierls bracket**, which is a linear response interpretation of bracket in covariant phase space formalism
- With Peierls bracket, we study the commutator between a **gravitational dressing** operator with the ADM energy operator in Jackiw-Teitelboim gravity

Outline

1. Introduction
2. Peierls bracket
3. Area operator A
4. Gravitational dressing operator

Diffeomorphism symmetries

- How to deal with diffeomorphism symmetries is a complicated problem in gravity
- In low energy limit, gravity can be described by an effective field theory with diffeomorphism symmetries
- Diffeomorphism symmetries → Gauge symmetries
- Gauge invariant observable:
  scattering amplitude,
  gauge invariant operators

- The scattering amplitude is not enough
- To understand physics behind event horizon, we need to study the operators supported deep into there

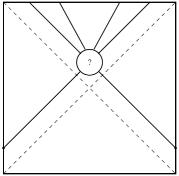
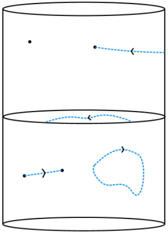


Figure: Two sides AdS-wormhole

Gauge invariant operators in QED

- QED is a well understood example with gauge symmetries
  Harlow 1510.07911

- Gauge invariant operators:
  local operators: $F_{\mu\nu}F^{\mu\nu}$
  non-local operators:
  $\exp[i\oint A_\mu dx^\mu]$
  $\bar{\psi}(x_1)\exp[\int_{x_2}^{x_1} ieA_\mu dx^\mu]\psi(x_2)$
  $\bar{\psi}(x_1)\exp[\int_{\partial}^{x_1} ieA_\mu dx^\mu]$



- Charged matter fields are associated with gauge fields (If we create charged particles, it should be always associated with the gauge fields to satisfy the constraint equations)

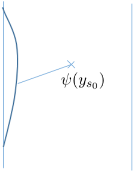
Diffeomorphism invariant operators in gravity

- Gravitational dressing: the matter fields are also associated with gravitational fields

  Giddings, Kinsella 1802.01602

- Relational construction: locating the operator $\psi(x)$ in a covariant way

- Example:
  Shoot geodesic from the spatial boundary
  with given distance y_{s_0}
  measure the matter fields $\psi(y_{s_0})$ at the end
  point of geodesic



  - (Any gauge invariant functional of configuration can be interpreted as a gauge invariant operator)
  - $\psi(y_{s_0})$ is a non-local operator

Motivations

- Fire-wall argument: creating a particle behind the event horizon decrease the ADM energy $[H, a_i^\dagger] = -w_i a_i^\dagger$

  Marolf, Polchinski 1307.4706

  The gravitational dressing is both necessary and complicated: non-zero commutator with ADM energy operator
  Higher order terms with respect to $\sqrt{G_N}$
- Bulk reconstruction

  Almheiri, Dong, Harlow 1411.7041

- In this work, we study the **gravitational dressing** operator in **classical limit**
- Classical limit:
  System $\rightarrow$ phase space
  $[\cdot,\cdot] \rightarrow \{\cdot,\cdot\}$
- In the following, we study the **Hamiltonian flow** of the gravitational dressing operator, from which we can directly read out the brackets between the gravitational dressing operator with any other operators for example the ADM energy operator
- The **Peierls bracket** gives a convenient way to study the Hamiltonian flow in covariant phase space formalism

Peierls bracket

- The Peierls bracket is a linear response like computation of gauge invariant operators' bracket in covariant phase space formalism Harlow, Wu 1906.08616
- (Inspiration from quantum mechanics:
$$\begin{aligned}\Delta O &= \langle \Omega \mid e^{i\lambda A} O e^{-i\lambda A} \mid \Omega \rangle - \langle \Omega \mid O \mid \Omega \rangle \\ &= i\lambda \langle \Omega \mid [A, O] \mid \Omega \rangle + \mathcal{O}(\lambda^2)\ )\end{aligned}$$
- In classical mechanics:
$\{f, g\}$ can be interpreted as the linear response of quantity f to the deformation of the action by $-g$

Symplectic geometry

- To describe the formalism, we need to review the symplectic geometry and covariant phase space formalism
- Symplectic geometry:

  Phase space: abstract manifold $\mathcal{P}$

  Symplectic form: closed non-degenerate two form Ω

  $\Omega = \frac{1}{2}\Omega_{ab} dy^a \wedge dy^b \quad (\Omega_{ab}$ is reversible)

  Ω_{ab} is an anti-symmetric (0,2)-type tensor
- Poisson bracket: $\{f, g\} = \Omega^{-1}(\delta f, \delta g) = (\Omega^{-1})^{ab} \frac{\partial f}{\partial y^a} \frac{\partial g}{\partial y^b}$
- Compare with standard Hamiltonian mechanics:

  Choose coordinate such that $\Omega = \sum_i \delta p_i \wedge \delta x^i$

  Bracket: $\{f, g\} = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i}$

- The Hamiltonian flow is an important notion in symplectic geometry
- Operator $f \leftrightarrow$ Hamiltonian flow X_f
  Hamiltonian equation:
  $X_f(g) = \Omega^{-1}(\delta g, \delta f) = \{g, f\}$
  or $\delta f = \Omega(\cdot, X_f) = -X_f \cdot \Omega$
  (old version: $\frac{d}{dt} g = \{g, H\}$)
- The Poisson bracket can be directly read out by acting the Hamiltonian flow of one operator onto another operator
  $\{g, f\} = X_f(g)$

Covariant phase space

- The symplectic geometry can be realized in covariant phase space formalism
- Pre-phase space: set of solutions $\tilde{\mathcal{P}}$
  Pre-symplectic form $\tilde{\Omega}$: closed but may be degenerate
  Zero modes of $\tilde{\Omega} \leftrightarrow$ gauge symmetries
- Quotient by zero modes: $\tilde{\mathcal{P}} \to \mathcal{P} \quad \tilde{\Omega} \to \Omega$
- In principle the Hamiltonian flow and the bracket can be computed with the phase space $\mathcal{P}$ and symplectic form Ω after we reduce the gauge redundancies

Hamiltonian flow and bracket

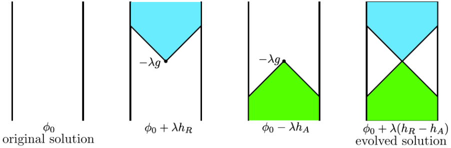
- We can also keep the gauge redundancies and do computation in pre-phase space $\tilde{\mathcal{P}}$ and with pre-symplectic form $\tilde{\Omega}$
- For gauge invariant operators f g
  Hamiltonian flow: $\delta f = -\tilde{X}_f \cdot \tilde{\Omega}$
  Bracket: $\{g, f\} = \tilde{X}_f(g)$
- Ambiguities in $\tilde{X}_f \leftrightarrow$ zero modes of $\tilde{\Omega} \leftrightarrow$ gauge symmetries
- $\{g, f\}$ is unique
- The Hamiltonian flow $\tilde{X}_f$ is the crucial quantity, from which we can directly read out the bracket
- The Peierls bracket gives a convenient way to compute $\tilde{X}_f$ in pre-phase space
- (In the original paper, the Peierls bracket is indeed a computation for bracket like $\{f, g\}$. However, in our set up, we refer the Peierls bracket as the computation of Hamiltonian flow X_g. The bracket can be directly read out by acting the Hamiltonian flow onto the other quantity $X_g f = \{f, g\}$)

Peierls bracket

- To get the Hamiltonian flow $\tilde{X}_f$, we consider a deformed theory with action $S = S_0 - kf$
- Solutions: $\phi^a = \phi_0^a + kh^a + \mathcal{O}(k^2)$
  Retarded solution: $h_R^a|_{t=-\infty} = 0$
  Advanced solution: $h_A^a|_{t=\infty} = 0$
- $h_R^a - h_A^a$ is a linear solution of the original theory:
  $\tilde{X}_f = \int d^d x (h_R^a(x) - h_A^a(x)) \frac{\delta}{\delta \phi^a(x)}$
  $\tilde{X}_f \cdot \delta E_a = 0 \quad (E_a$ is the e.o.m of the original theory)
- $\tilde{X}_f$ is the Hamiltonian flow of operator f

Harlow, Wu 1906.08616

- Evolving forward and backward interpretation



- When the operator only supports at $t = 0$, the original solution and the evolved solution can be described by the initial values at $t = 0^-$ and $t = 0^+$

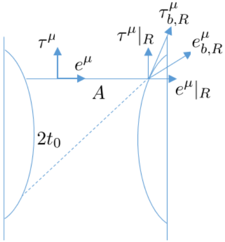
Simple example: point particle

- Action: $S_0 = \int_{t_i}^{t_f} dt \frac{1}{2}\dot{x}^2$;
  Question: Hamiltonian flow for operator $x(t=0)$
- Solutions: $x(t) = x_0 + p_0 t$
  Symplectic form: $\omega = \delta p_0 \wedge \delta x_0$
  The Hamiltonian flow: $X_{x(0)} = -\frac{\delta}{\delta p_0}$
- Peierls bracket:
  Deformed action: $S = S_0 - kx(0)$
  Taking a variation: $\delta S = \int dt(-\ddot{x}(t) - \delta(t))\delta x(t)$
  Solutions: $x(t) = \left\{ \begin{array}{cc} x_0 + p_0 t & t < 0 \\ x_0 + (p_0 - k)t & t > 0 \end{array} \right.$
  Hamiltonian flow: $X_{x(0)} = -\frac{\delta}{\delta p_0}$

Hamiltonian flow for area operator X_A

- Pure Jackiw-Teitelboim gravity
  $S_0 = \int d^2x\sqrt{-g}\Phi(R+2) + \int dt\sqrt{-\gamma}\Phi(2K-2)$
  with b.d $\Phi|_\partial = \frac{\phi_0}{\epsilon} \quad ds^2|_\partial = -\frac{dt^2}{\epsilon^2}$
- Question: The Hamiltonian flow for the area operator X_A

- Phase space: (A, η)
  where $\tau^\mu_{b,R} = \cosh\eta\tau^\mu|_R + \sinh\eta e^\mu|_R$
  $\Omega = \frac{2\phi_0}{\epsilon}\frac{\tanh A/2}{\cosh^2\eta}\delta\eta\wedge\delta A$



- Hamiltonian flow: $X_A = -\frac{\epsilon}{2\phi_0}\frac{\cosh^2\eta}{\tanh A/2}\frac{\delta}{\delta\eta}$

- Peierls bracket computation:
- Deformed action $S = S_0 - kA$
- e.o.m: $R + 2 = 0$

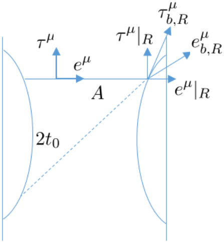
$$(\nabla^{\mu}\nabla^{\nu}\Phi - g^{\mu\nu}\nabla^2\Phi + g^{\mu\nu}\Phi) - \frac{1}{\sqrt{-g}}\frac{\delta A}{\delta g_{\mu\nu}} = 0$$

- Operator variation: $\delta A = \int ds \frac{1}{2} e^{\mu} e^{\nu} \delta g_{\mu\nu}(y(s))$
  geodesic parametrization: $y^{\mu}(s)$
- ADM decomposition:

$$D^2\Phi - \Phi = 0$$

$$e^{\mu} D_{\mu}(\tau^{\nu}\nabla_{\nu}\Phi) = 0$$

$$\tau^{\mu}\Phi|_{\Sigma_+} - \tau^{\mu}\Phi|_{\Sigma_-} = \frac{k}{2} \quad \Phi|_{\Sigma_+} - \Phi|_{\Sigma_-} = 0$$



  - Solutions:

$$\Phi = \frac{\phi_0}{\epsilon}\frac{\cosh[s-A/2]}{\cosh[A/2]} \qquad \tau^{\mu}\nabla_{\mu}\Phi = -\frac{\phi_0}{\epsilon}\tanh\eta\tanh\frac{A}{2}$$

$$A|_{\Sigma_+} - A|_{\Sigma_-} = 0$$

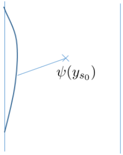
$$\eta|_{\Sigma_+} - \eta|_{\Sigma_-} = -k\frac{\epsilon}{2\phi_0}\cosh^2\eta\coth\frac{A}{2}$$

  - Hamiltonian flow: $X_A = -\frac{\epsilon}{2\phi_0}\frac{\cosh^2\eta}{\tanh A/2}\frac{\delta}{\delta\eta}$

Gravitational dressing

- Jackiw-Teitelboim gravity coupled with a matter field $S_0 = \int d^2x\sqrt{-g}[\Phi(R+2) - \frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi]$
- Initial values: induced metric $\gamma_{\mu\nu}$, the extrinsic curvature $K_{\mu\nu}$, the dilaton field Φ and its time derivative $\tau^\mu\nabla_\mu\Phi$, the matter field ψ and its time derivative $\tau^\mu\nabla_\mu\psi$; the relative angle with the spatial boundaries
- Constraint equations: $D^2\Phi - K\tau^\mu\nabla_\mu\Phi - \Phi + \frac{1}{4}(\tau^\mu\nabla_\mu\psi)^2 + \frac{1}{4}\sigma^{\mu\nu}D_\mu\psi D_\nu\psi = 0$ $D_\rho(\tau^\nu\nabla_\nu\Phi) - KD_\rho\Phi + \frac{1}{2}D_\rho\psi\tau^\nu\nabla_\nu\psi = 0$

- Gravitational dressing: $\psi(y_{s_0})$
  $y^\mu(s)$: geodesic orthogonal to the boundary
  $\frac{d^2}{ds^2}y^\mu(s)+\Gamma^\mu_{\nu\rho}(y(s))\frac{dy^\nu}{ds}\frac{dy^\nu}{ds}=0$



- Question: Hamiltonian flow $X_{\psi(y_{s_0})}$
  - $y^\mu(s)$ is a functional of the metric
    Taking variation with respect to metric:
    $\frac{D^2}{ds^2}\delta y^\mu+R_{\lambda\nu;}{}^{\mu}{}_{\rho}\delta y^\lambda\frac{dy^\nu}{ds}\frac{dy^\rho}{ds}+\delta_g\Gamma^\mu_{\nu\rho}(y(s))\frac{dy^\nu}{ds}\frac{dy^\rho}{ds}=0$
  - $\delta y^\mu=\alpha(s)e^\mu+\beta(s)\tau^\mu$
    with
    $\alpha(s)=\int_0^s ds_1(-\frac{1}{2})\delta g_{\mu\nu}(y(s_1))e^\mu(s_1)e^\nu(s_1)$
    $\beta(s)=\int_0^s ds_1\cosh(s-s_1)\delta\mathbf{g}_{\mu\nu}(\mathbf{y}(\mathbf{s_1}))\tau^\mu(\mathbf{s_1})\mathbf{e}^\nu(\mathbf{s_1})$
    $\qquad-\int_0^s ds_1\frac{l}{2}\sinh(s-s_1)\tau^\sigma(\mathbf{s_1})\nabla_\sigma\delta\mathbf{g}_{\mu\nu}(\mathbf{y}(\mathbf{s_1}))\mathbf{e}^\mu(\mathbf{s_1})\mathbf{e}^\nu(\mathbf{s_1})$

- Deformed action: $S = S_0 - k\psi(y_{s_0})$
- e.o.m

  $R + 2 = 0$

  $\sqrt{-g}\nabla^2\psi - k\delta^2(x - y_{s_0}) = 0$

  $$(\nabla^\mu\nabla^\nu\Phi - g^{\mu\nu}\nabla^2\Phi + \Phi g^{\mu\nu} \\ + \frac{1}{2}\nabla^\mu\psi\nabla^\nu\psi - \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi) - k\frac{1}{\sqrt{-g}}\frac{\delta\psi(y_{s_0})}{\delta g_{\mu\nu}} = 0$$

  - Solutions:

    $\psi|_{\Gamma_+} = \psi|_{\Gamma_-}$

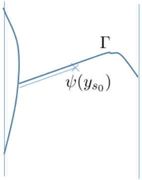
    $\tau^\mu\partial_\mu\psi|_{\Gamma_+} = \tau^\mu\partial_\mu\psi|_{\Gamma_-} - k\delta(s - s_0)$

    for matters;

    $$\begin{cases} \Phi|_{\Gamma_+} = \Phi|_{\Gamma_-} + \frac{k}{2}\tau^\mu(s_0)\partial_\mu\psi(y_{s_0})\sinh(s_0 - s) \\ \tau^\rho\partial_\rho\Phi|_{\Gamma_+} = \tau^\rho\partial_\rho\Phi|_{\Gamma_-} - \frac{k}{2}e^\mu(s_0)\partial_\mu\psi(y_{s_0}) \end{cases} \quad \text{for } s < s_0$$

    $$\begin{cases} \Phi|_{\Gamma_+} = \Phi|_{\Gamma_-} \\ \tilde{\tau}^\mu\partial_\mu\Phi|_{\Gamma_+} = \tilde{\tau}^\mu\partial_\mu\Phi|_{\Gamma_-} \end{cases} \quad \text{for } s > s_0$$

    for dilaton fields



- Solutions:

  $$\psi|_{\Gamma_+} = \psi|_{\Gamma_-}$$

  $$\tau^\mu \partial_\mu \psi|_{\Gamma_+} = \tau^\mu \partial_\mu \psi|_{\Gamma_-} - k\delta(s - s_0)$$

  for matters;

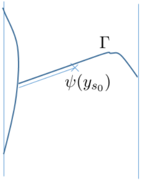
  $$\begin{cases} \Phi|_{\Gamma_+} = \Phi|_{\Gamma_-} + \frac{k}{2}\tau^\mu(s_0)\partial_\mu\psi(y_{s_0})\sinh(s_0 - s) \\ \tau^\rho\partial_\rho\Phi|_{\Gamma_+} = \tau^\rho\partial_\rho\Phi|_{\Gamma_-} - \frac{k}{2}e^\mu(s_0)\partial_\mu\psi(y_{s_0}) \end{cases} \quad \text{for } s < s_0$$

  $$\begin{cases} \Phi|_{\Gamma_+} = \Phi|_{\Gamma_-} \\ \tilde{\tau}^\mu\partial_\mu\Phi|_{\Gamma_+} = \tilde{\tau}^\mu\partial_\mu\Phi|_{\Gamma_-} \end{cases} \quad \text{for } s > s_0$$

- Some comments:

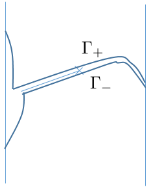
  1. The metric is still AdS_2
  2. The discontinuity only appears on the geodesic
  3. The dilaton field is not continuous
  4. The configuration satisfies the constraints
  5. $\Delta\Phi(s = 0) \neq 0$

  (The boundary jumps along the normal direction)



- Hamiltonian flow (for initial values):

  1. Shoot geodesic and find the end points
  2. Extend the geodesic to the whole Cauchy surface Γ_-
  3. The metric is still AdS_2
  4. Solve the matter fields $(\psi,\ \tau^\mu\nabla_\mu\psi)$ from source term
  5. Solve the dilaton field by constraints
  6. Allow the boundary jump along normal direction to satisfy boundary condition $\Phi|_\partial = \frac{\phi_0}{\epsilon}$



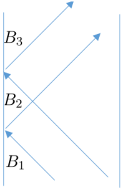
Bracket

- To better understand the Hamiltonian flow, we act the Hamiltonian flow onto other gauge invariant operators, which gives their bracket
- Bracket with ADM energy operator
  $\{H_L, \psi_{s_0}\} = X_{\psi_{s_0}} H_L = \tau^\mu(s_0)\partial_\mu\psi(y_{s_0})[\frac{1}{4\phi_0} e^{\tilde{s}_0} H_L - e^{-\tilde{s}_0}]$
  with $s_0 = -\log\epsilon + \tilde{s}_0$
- The result is consistent with Schwarzian theory's computation

Smearing with coherent wave packet

- Operators: $\psi(y(s_0))$, $\tau^\mu \partial_\mu \psi(y(s_0))$
- $\psi^\dagger_{s_0,k}$: Creation operator which create a wave packet with center of location s_0 and center of momentum k
- $[H_L, \psi^\dagger_{s_0,k}] \to i\{H_L, \psi^\dagger_{s_0,k}\} \approx -k[\frac{1}{4\phi_0} e^{\tilde{s}_0} H_L - e^{-\tilde{s}_0}]\psi^\dagger_{s_0,k}$
- The coefficient $-k[\frac{1}{4\phi_0} e^{\tilde{s}_0} H_L - e^{-\tilde{s}_0}]$ gives the energy change after creating the particle

- $[H_L, \psi^\dagger_{s_0,k}] \to i\{H_L, \psi^\dagger_{s_0,k}\} \approx -k[\frac{1}{4\phi_0} e^{\tilde{s}_0} H_L - e^{-\tilde{s}_0}]\psi^\dagger_{s_0,k}$
- $e^{2\tilde{s}_0} < \frac{4\phi_0}{H_L}$ H_L increase
  $e^{2\tilde{s}_0} > \frac{4\phi_0}{H_L}$ H_L decrease



- (Assuming the background is multi-shock wave configuration)
  The critical point is at the event horizon
- (The configurations are the same at B_1, B_2, B_3
  $\Phi = a\frac{\cos\tau}{\sin\sigma} + b\frac{\sin\tau}{\sin\sigma} + c\frac{\cos\sigma}{\sin\sigma}$)
  - Marolf-Polchinski firewall argument Marolf, Polchinski 1307.4706
    Typicality, gravitational dressing, quantum correction

Conclusion

- In this work, we develop the Peierls bracket in covariant phase space formalism
- We study the Hamiltonian flow of gravitational dressing operator and also study the bracket with other operators
- Up to now, we only deal with simple examples. Hope this formalism will have more applications in future

Thanks

Thanks for your attention!