Complete prepotential for five dimensional $\mathcal{N}$=1 superconformal field theory

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Based on the work with Hirotaka Hayashi, Sung-Soo Kim, and Kimyeong Lee [ArXiv:1912.10301]

5d (supersymmetric) gauge theory

Perturbatively non-renormalizable

(gauge coupling constant has negative mass dimension)

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5d SUSY gauge theory should be understood as SCFT + deformation

UV	RG flow	IR
5d SCFT	Relevant deformation →	5d SUSY gauge theory

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UV	RG flow	IR
5d SCFT	Relevant deformation	5d SUSY gauge theory

Assume the that non-trivial UV fixed point exist.

(The SCFT is uniquely determined.)

(c.f. Non-trivial UV fixed point for 5d N=1 SU(2) gauge theory. [Seiberg ’96])

UV

RG flow

IR

5d SCFT → 5d SUSY gauge theory

Relevant deformation

Deformation parameters:

m_0 → Gauge coupling constant g
(Mass of instanton particle)

$\left(m_0 = \frac{1}{g^2}\right)$

m_f → Hypermultiplet masses

$m_0 > 0$ is a necessary condition to obtain the gauge theory

Does it make sense to consider the parameter region

$$m_0 = \frac{1}{g^2} < 0?$$

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$$m_0 = \frac{1}{g^2} < 0?$$

YES in the following sense

RG flow

IR

UV

5d SCFT

Relevant deformation e.g. $m_0 > 0$ → **5d SYM**

UV duality

Relevant deformation e.g. $m_0 < 0$ → **Another 5d SYM or Non-Lagrangian theory**

We would like to compute prepotential of the SCFT defined for the whole parameter region of the deformation parameters

"Complete" prepotential

$$F(a_i, m_0, m_f)$$

a_i: Coulomb moduli (vector multiplet)

m_0, m_f: deformation parameters

$$-\infty < m_0 < \infty, \qquad -\infty < m_f < \infty,$$

Property of complete prepotential

0. Defined for the whole parameter region

1. Reproduce the IMS prepotential [Intriligator, Morrison, Seiberg '97]

- Agree with IMS Prepotential inside the parameter region for the gauge theory.
- Additional corrections may appear outside.

2. Consistent with the 5-brane web

$$\frac{\partial F}{\partial \phi_i} = (\text{Monopole tension}) = (\text{Area of D3-brane in the web})$$

3. Respect the global symmetry

Invariant under the Weyl group of the global symmetry of the SCFT.

4. Consistent with UV-duality

Reproduce two or more different IMS prepotentials depending on the parameter region.

5. Derived from the Nekrasov partition function.

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5d $\mathcal{N}$=1 gauge theory at Coulomb phase

"IMS Prepotential" [Intriligator, Morrison, Seiberg '97]

- **1-loop exact**
- **Locally cubic due to gauge invariance**
- **Coefficients of the polynomial changes depending on the parameter region.**

IMS Prepotential [Intriligator, Morrison, Seiberg '97]

$$F_{\text{IMS}} = F_{\text{classical}} + F_{\text{1-loop}}$$

$$F_{\text{classical}} = \frac{1}{2} m_0 h_{ij} \phi_i \phi_j + \frac{k}{6} d_{ijk} \phi_i \phi_j \phi_k$$

Gauge kinetic term

Chern-Simons term

ϕ_i : Vector multiplet, $m_0 = \frac{1}{g^2}$: Bare coupling

$h_{ij} = \mathrm{Tr}(T_i T_j)$, $d_{ijk} = \frac{1}{2} \mathrm{Tr}\left(T_i (T_j T_k + T_k T_j)\right)$,

k : Chern-Simons level

IMS Prepotential [Intriligator, Morrison, Seiberg '97]

$$F_{\text{IMS}} = F_{\text{classical}} + F_{\text{1-loop}}$$

$$F_{\text{1-loop}} = \frac{1}{12}\sum_{\mathbf{R}} |\mathbf{R}\cdot\phi|^3 - \frac{1}{12}\sum_{f}\sum_{\mathbf{w}\in\mathbf{W}_f} |\mathbf{w}\cdot\phi + m_f|^3$$

Vector multiplet contribution

Hypermultiplet contribution

R : Root of the gauge group, W_f : Weight of the gauge group

f : Label for the hypermultiplet, m_f : Mass of the hypermultiplet

Observation

$$F_{\text{1-loop}} \sim \sum (\text{BPS mass})^3$$

$$\text{BPS mass} = \left| \sum_i n_i a_i + \sum_f n_f m_f \right|$$

$a_i = \langle \phi_i \rangle$: Coulomb moduli

m_f : Hypermultiplet mass

n_i : electric charge

n_f : flavor charge

Coefficients of the polynomial changes at the place where a BPS particle becomes massless

Digression: BPS formula for 4D and 5D

4D theory:

Instanton is a 0 dim object ,
Magnetic monopole is a 1dim object (particle)

$$\text{BPS particle mass} = |n_e a + \underline{n_m a_D} + n_f m_f|$$

n_e : electric charge, n_m : magnetic charge, n_f : flavor charge

5D theory:

Instanton is a 1 dim object (particle),
Magnetic monopole is a 2 dim object (string)

$$\text{BPS particle mass} = |n_e a + \underline{n_0 m_0} + n_f m_f|$$

n_e : electric charge, n_0 : instanton charge, n_f : flavor charge

New contributions in complete prepotential

$$\sum (\text{BPS mass})^3$$

$$\text{BPS mass} = \left| \sum_i n_i a_i + \underline{n_0 m_0} + \sum_f n_f m_f \right|$$

BPS particle with non-zero instanton charge may also contribute when it becomes massless

Convention

1. **Choose one Weyl chamber of the gauge group**
   (Absolute value for vector multiplet can be removed)

2. **Rewrite the absolute value according to the identity**

$$|x| = x - 2 \llbracket x \rrbracket \qquad \llbracket x \rrbracket := x\,\theta(-x) = \begin{cases} 0 & (x \geq 0) \\ x & (x < 0) \end{cases}$$

c.f. [Closset, del Zotto, Saxena '18]

$$F_{\text{Complete}} = F_{\text{IMS}} + \frac{1}{6} \sum_{\substack{n_i, n_f \\ n_0 > 1}} \llbracket \sum_i n_i a_i + n_0 m_0 + \sum_f n_f m_f \rrbracket^3$$

Vanishes if $m_0 \gg |a_i|, |m_f|$

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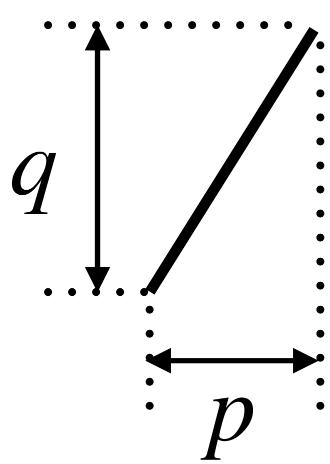
(p, q) 5-brane web diagram

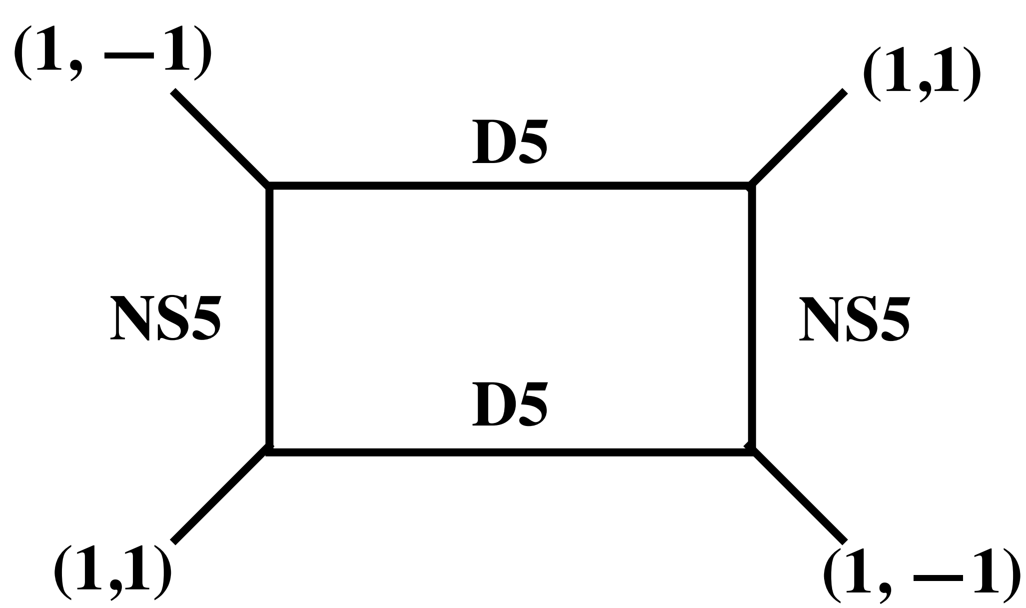
[Aharony, Hanany '97]
[Aharony, Hanany, Kol '97]

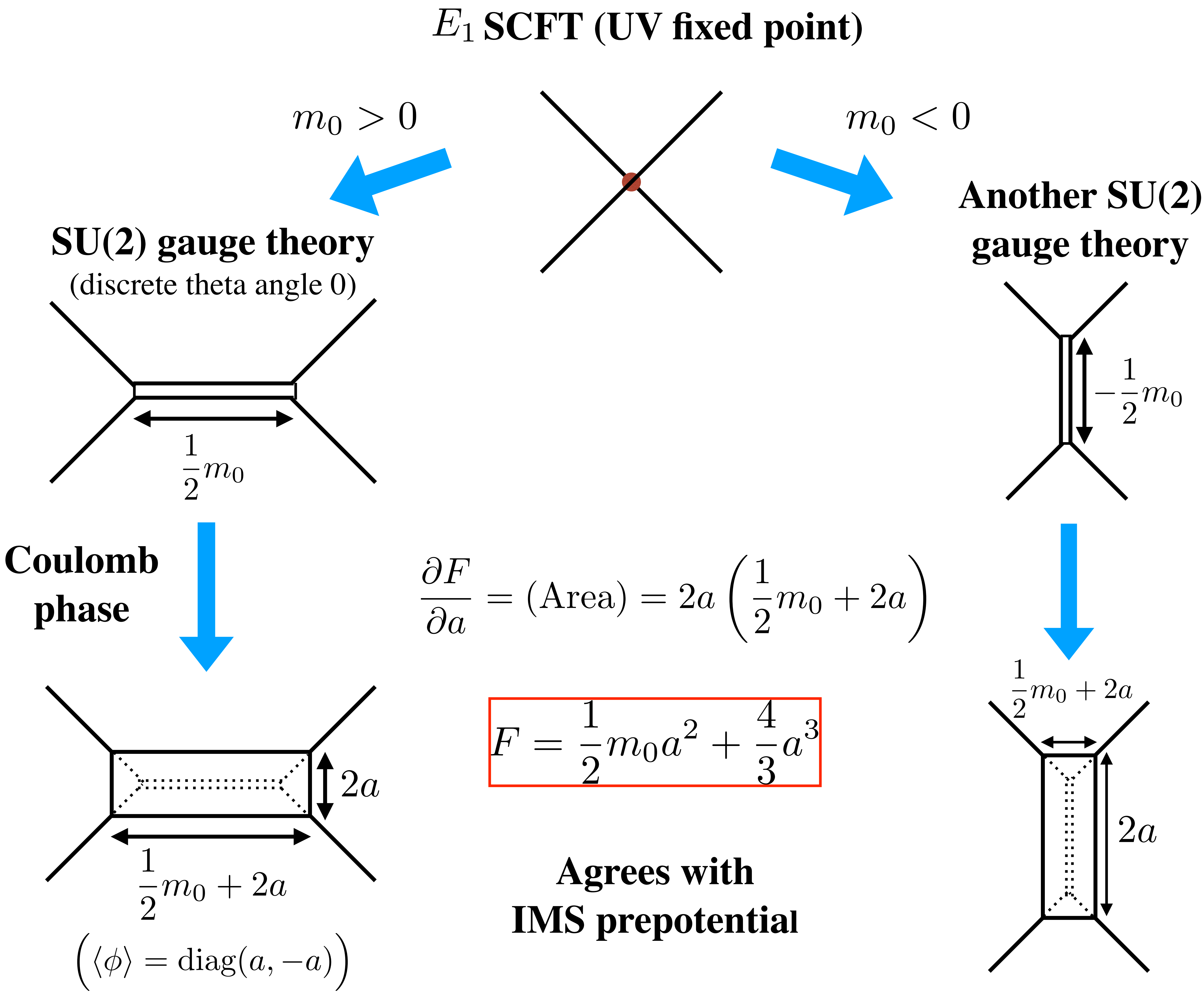
	0	1	2	3	4	5	6	7	8	9
5-brane	—	—	—	—	—	web		·	·	·



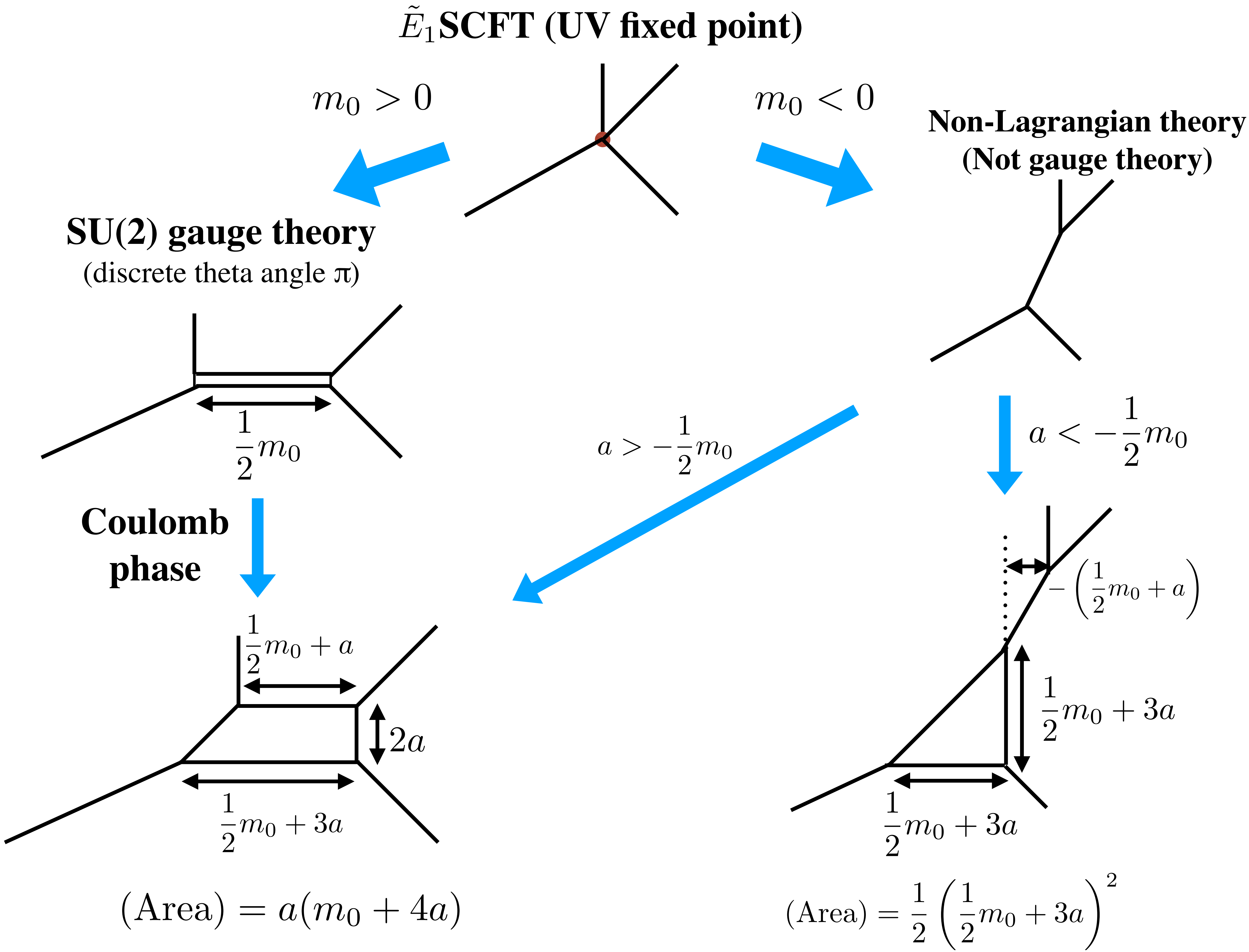
(p,q) 5-brane = p D5-brane + q NS5-brane

(1,0) 5-brane = D5 brane
(0,1) 5-brane = NS5 brane





E_1 SCFT (UV fixed point)
$m_0 > 0$
$m_0 < 0$
SU(2) gauge theory
(discrete theta angle 0)
Another SU(2) gauge theory
$\frac{1}{2}m_0$
$-\frac{1}{2}m_0$
Coulomb phase
$\frac{\partial F}{\partial a} = (\text{Area}) = 2a\left(\frac{1}{2}m_0 + 2a\right)$
$F = \frac{1}{2}m_0 a^2 + \frac{4}{3}a^3$
Agrees with IMS prepotential
$2a$
$\frac{1}{2}m_0 + 2a$
$(\langle\phi\rangle = \mathrm{diag}(a, -a))$
$\frac{1}{2}m_0 + 2a$
$2a$



$\tilde{E}_1$SCFT (UV fixed point)
$m_0 > 0$
$m_0 < 0$
Non-Lagrangian theory
(Not gauge theory)
SU(2) gauge theory
(discrete theta angle π)
$\frac{1}{2}m_0$
$a > -\frac{1}{2}m_0$
$a < -\frac{1}{2}m_0$
Coulomb phase
$\frac{1}{2}m_0 + a$
$2a$
$\frac{1}{2}m_0 + 3a$
$-\left(\frac{1}{2}m_0 + a\right)$
$\frac{1}{2}m_0 + 3a$
$\frac{1}{2}m_0 + 3a$
$(\text{Area}) = a(m_0 + 4a)$
$(\text{Area}) = \frac{1}{2}\left(\frac{1}{2}m_0 + 3a\right)^2$

Complete prepotential for $\tilde{E}_1$ theory

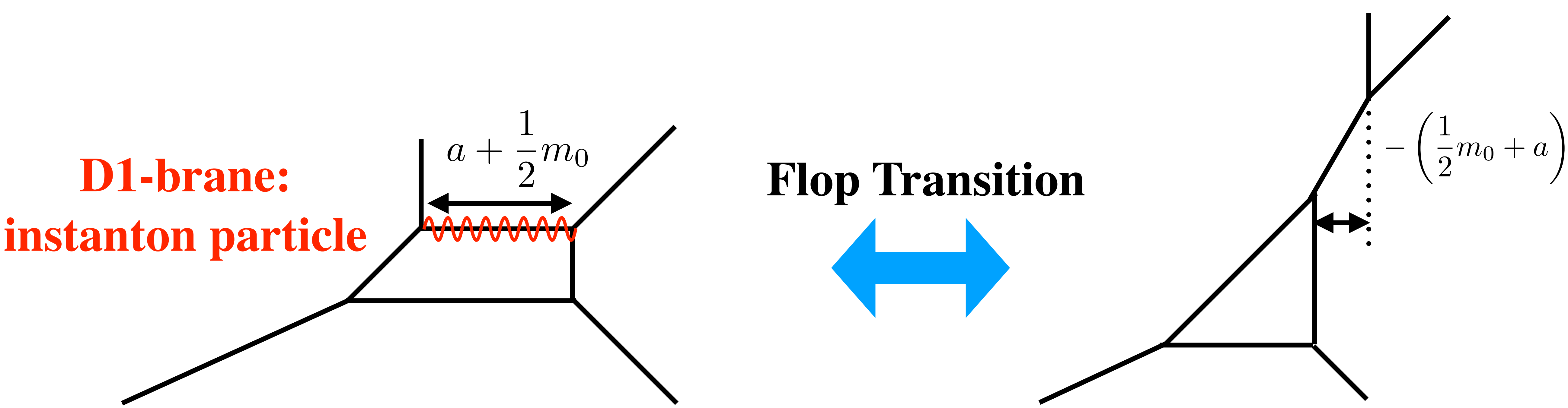
c.f. [Morrison, Seiberg '97]

$$F = \frac{1}{2}m_0 a^2 + \frac{4}{3}a^3 + \frac{1}{6}[\![a + \frac{1}{2}m_0]\!]^3 + (m_0^3 \text{ terms}) \qquad a \geq 0$$

IMS prepotential

Contribution from "instanton particle"

(BPS particle with non-zero instanton charge at Coulomb phase)



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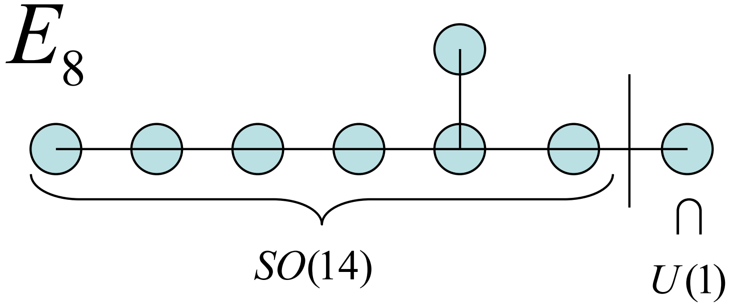
Global symmetry at UV fixed point for SU(2) gauge theory with N_f flavor

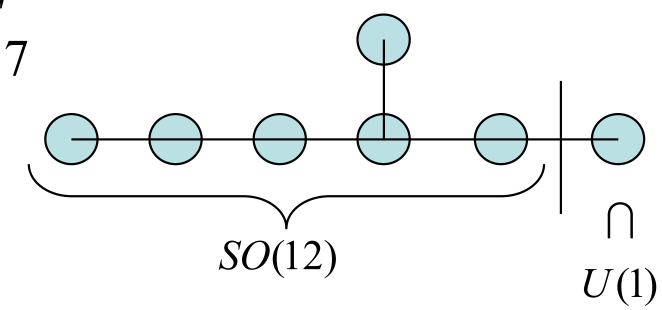
[Seiberg '96]

$$SO(2N_f) \times U(1)_I \subset E_{N_f+1}$$

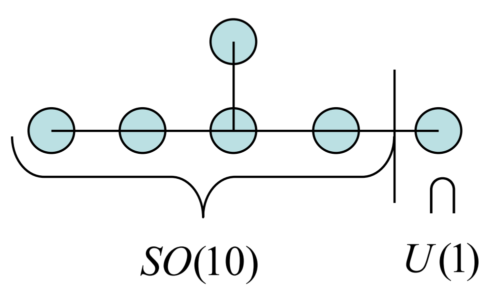
E_8



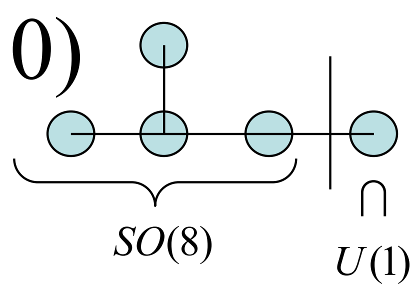
E_7



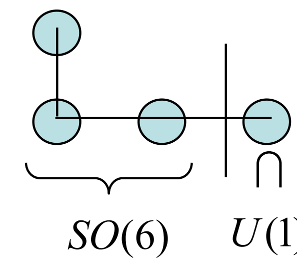
E_6



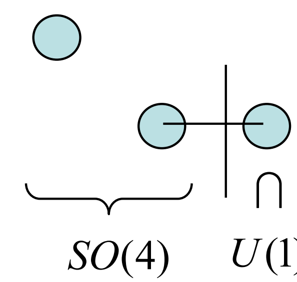
$E_5 = SO(10)$



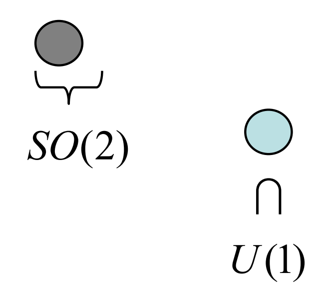
$E_4 = SU(5)$



$E_3 = SU(2) \times SU(3)$

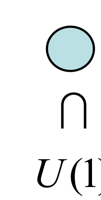


$E_2 = U(1) \times SU(2)$



$E_1 = SU(2)$

$\tilde{E}_1 = U(1)$



How can we see the global symmetry $E_1 = SU(2)$ from prepotential?

Weyl reflection of E_1

[Aharony, Hanany, Kol '97]

$$m_0 \to -m_0$$
$$a \to a + \frac{1}{4} m_0$$

"Invariant Coulomb branch parameter": $\tilde{a} = a + \frac{1}{8} m_0$

[Mitev, Pomoni, Taki, Yagi '14]

$$\begin{aligned} F(a) &= \frac{1}{2} m_0 a^2 + \frac{4}{3} a^3 \\ &= \frac{4}{3} \tilde{a}^3 - \frac{1}{2} {m_0}^2 \tilde{a} + \frac{1}{96} {m_0}^3 \end{aligned}$$

Prepotential is invariance under the Weyl reflection of E_1

SU(2) with $N_f = 1$ flavor, $E_2 = SU(2) \times U(1)$

Weyl Reflection of E_2: $(x, y, \tilde{a}) \to (-x, y, \tilde{a})$

$$x := \frac{1}{4}m_0 + \frac{1}{4}m_1, \quad y := -\frac{1}{4}m_0 + \frac{7}{4}m, \quad \tilde{a} = a + \frac{1}{7}m_0$$

$$F = \frac{7}{6}\tilde{a}^3 - \left(x^2 + \frac{1}{7}y^2\right)\tilde{a} + \frac{1}{6}[\![\, \tilde{a} + \frac{4}{7}y \,]\!]^3 + \frac{1}{6}[\![\, \tilde{a} + x - \frac{3}{7}y \,]\!]^3$$

IMS prepotential

$$[\![\, a \pm m \,]\!]^3$$

$$+\frac{1}{6}[\![\, \tilde{a} - x - \frac{3}{7}y \,]\!]^3$$

Contribution from instanton particle

$$[\![\, a + \frac{1}{2}(m_0 - m) \,]\!]^3$$

Agree with the complete prepotential computed from the area

[H.Hayashi, S.S.Kim, K.Lee, F.Yagi '17]
c.f. [Morrison, Seiberg '97]

Complete Prepotential for 5d rank 1 SCFT

(SU(2) with N_f flavor)

$$F = \frac{8-N_f}{6}\tilde{a}^3 + \left(\frac{1}{8-N_f}{m_0}^2 + \sum_{k=1}^{N_f}{m_k}^2\right)\tilde{a} + \frac{1}{6}\sum_{\mathbf{w}\in\text{weight of }E_{N_f+1}} \left\| \tilde{a} + \mathbf{w}\cdot\mathbf{m} \right\|^3$$

$$\tilde{a} := a + \frac{1}{8-N_f}m_0$$

e.g. $N_f = 7$

$$F = \frac{1}{6}\tilde{a}^3 - \frac{1}{2}\sum_{k=0}^{7}{m_k}^2\tilde{a} + \frac{1}{6}\sum_{\substack{\{s_i=\pm1\}\\ \sum_i x_i=0 \bmod 4}} \left\| \tilde{a} + \frac{1}{2}\sum_{k=0}^{7}s_k m_k \right\|^3$$

$$+ \frac{1}{6}\sum_{\substack{s_1=\pm1\\ s_2=\pm1}}\sum_{0\leq i<j\leq 7}\left\| \tilde{a} + s_1 m_i + s_2 m_j \right\|^3 + \frac{1}{6}\left\| \tilde{a} \right\|^3 \times 8$$

248 terms

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Example of UV-duality (triality)

Rank 2 SCFT with SO(20) global symmetry

→ **Sp(2) + 9 flavor**

→ **$SU(3)_{1/2}$ + 9 flavor**

→ **[4]—SU(2)—SU(2)—[3]**

[D. Gaiotto, H. Kim '15]
[H.Hayashi, S-S.Kim, K. Lee, F.Yagi '15, '16]
[Y.Yun '16]
[P. Jefferson, S. Katz, H. Kim, C. Vafa '18]

Complete Prepotential for the rank 2 SCFT with SO(20) global symmetry

$$
\begin{aligned}
F_{\text{Complete}} =& \frac{1}{6}(\tilde{a}_1^3 - 2\tilde{a}_2^3) + \tilde{a}_1\tilde{a}_2^2 - \frac{1}{2}\sum_{f=0}^{9} m_f^2(\tilde{a}_1+\tilde{a}_2) \\
&+ \frac{1}{6}\sum_{0\le i<j\le 9}\ \sum_{\substack{s_1=\pm1\\ s_2=\pm1}} \left\|\, \tilde{a}_1 + s_1 m_i + s_2 m_j \,\right\|^3 + \frac{1}{6}\sum_{i=0}^{9}\sum_{s=\pm1}\left\|\, \tilde{a}_2 + s\, m_i \,\right\|^3 \\
&+ \frac{1}{6}\sum_{\substack{\{s_i=\pm1\}\\ \sum_i s_i = 0 \bmod 4}} \left\|\, \tilde{a}_1+\tilde{a}_2+\frac{1}{2}\sum_{i=0}^{9} s_i m_i \,\right\|^3 \\
&+ \frac{1}{6}\sum_{0\le i_1<i_2<i_3<i_4<i_5\le 9}\ \sum_{\{s_k=\pm1\}} \left\|\, 2\tilde{a}_1+\tilde{a}_2+\sum_{k=1}^{5} s_k m_{i_k} \,\right\|^3 .
\end{aligned}
$$

$\tilde{a}_i \quad (i=1,2):$ invariant Coulomb moduli

$m_i:\quad SO(20)$ fugacity

Reduction to Sp(2) 9 flavor

$$m_0 = m_0^{Sp}, \qquad m_f = m_f^{Sp} \quad (f = 1, 2, \cdots, 9)$$

$$\tilde{a}_1 = a_1^{Sp} + m_0^{Sp}, \quad \tilde{a}_2 = a_2^{Sp} \qquad \left(a_1^{Sp} \geq a_2^{Sp} \geq 0\right)$$

If we choose the parameter region $m_0^{Sp} \gg |a_i^{Sp}|, |m_f^{Sp}|$

$$F_{\text{Complete}} \to \frac{1}{2} m_0 \left((a_1^{Sp})^2 + (a_2^{Sp})^2\right) + \frac{1}{6}\left((a_1^{Sp})^3 - (a_2^{Sp})^3\right)$$

$$+ a_1^{Sp}(a_2^{Sp})^2 - \frac{1}{2}\sum_{i=1}^{2}\sum_{f=1}^{9} m_f^{Sp}(a_i^{Sp})^2$$

$$+ \frac{1}{6}\sum_{i=1}^{2}\sum_{f=1}^{9} \left\| a_i^{Sp} - m_f^{Sp} \right\|^3$$

$$= F_{\text{IMS}}^{Sp(2)+9F}$$

Reduction to $SU(3)_{1/2}$ 9 flavor

$$m_0 = \frac{3}{4}m_0^{SU} + \frac{1}{4}\sum_{f=1}^{9} m_i^{SU}, \qquad m_f = m_f^{SU} + \frac{1}{4}\left(m_0^{SU} - \sum_{f=1}^{9} m_f^{SU}\right) \qquad (f = 1, 2, \cdots 9)$$

$$\tilde{a}_1 = a_1^{SU} + m_0^{SU}, \qquad \tilde{a}_2 = a_2^{SU} + \frac{1}{4}\left(m_0^{SU} - \sum_{f=1}^{9} m_f^{SU}\right) \qquad (a_1^{SU} > a_2^{SU} > a_3^{SU} = -a_1^{SU} - a_2^{SU})$$

If we choose the parameter region $m_0^{SU} \gg |m_f^{SU}|, |a_i^{SU}|$

$$\begin{aligned}
F_{\text{Complete}} \to & \frac{1}{6}\left((a_1^{SU})^3 - (a_2^{SU})^3\right) + a_1^{SU}(a_2^{SU})^2 \\
& + \frac{1}{2}m_0^{SU}\left((a_1^{SU})^2 + a_1^{SU}a_2^{SU} + (a_2^{SU})^2\right) \\
& - \frac{1}{2}\sum_{f=1}^{9} m_f^{SU} a_1^{SU} a_2^{SU} - \frac{1}{2}\sum_{f=1}^{9}\left(a_1^{SU} + a_2^{SU}\right)({m_f}^{SU})^2 \\
& + \frac{1}{6}\sum_{f=1}^{9}\left(\llbracket a_1^{SU} - m_f^{SU} \rrbracket^3 + \llbracket a_2^{SU} - m_f^{SU} \rrbracket^3 + \llbracket -a_3^{SU} + m_f^{SU} \rrbracket^3\right) \\
= & F_{\text{IMS}}^{SU(3)_{1/2}+9F}
\end{aligned}$$

Reduction to [4]—SU(2)$^{(1)}$—SU(2)$^{(2)}$—[3]

$$m_1=\frac{1}{2}(-m_1^{(1)}+m_2^{(1)}-m_3^{(1)}-m_4^{(1)}),\qquad m_2=\frac{1}{2}(m_1^{(1)}-m_2^{(1)}-m_3^{(1)}-m_4^{(1)}),$$

$$m_3=\frac{1}{2}(m_1^{(1)}+m_2^{(1)}-m_3^{(1)}+m_4^{(1)}),\qquad m_4=\frac{1}{2}(m_1^{(1)}+m_2^{(1)}+m_3^{(1)}-m_4^{(1)}),$$

$$m_5=\frac{1}{2}m_0^{(1)}-m_{\rm bif}\qquad m_6=\frac{1}{2}m_0^{(1)}+m_{\rm bif}$$

$$m_7=\frac{1}{2}(m_0^{(1)}+m_0^{(2)}-m_1^{(2)}+m_2^{(2)}+m_3^{(2)}),\quad m_8=\frac{1}{2}(m_0^{(1)}+m_0^{(2)}+m_1^{(2)}-m_2^{(2)}+m_3^{(2)}),$$

$$m_9=\frac{1}{2}(m_0^{(1)}+m_0^{(2)}-m_1^{(2)}-m_2^{(2)}-m_3^{(2)}),\quad m_0=\frac{1}{2}(m_0^{(1)}+m_0^{(2)}+m_1^{(2)}+m_2^{(2)}-m_3^{(2)}),$$

$$\tilde{a}_1=a^{(2)}+m_0^{(1)}+m_0^{(2)},\qquad \tilde{a}_2=a^{(1)}-a^{(2)}+\frac{1}{2}m_0^{(1)}$$

If we choose the parameter region $m_0^{(1)},m_0^{(2)}\gg|m_f^{(1)}|,|m_f^{(2)}|,|a^{(1)}|,|a^{(2)}|$

$$\begin{aligned}F_{\rm Complete}\to&\frac{1}{6}\left(2(a^{(1)})^3+5(a^{(2)})^3-6a^{(1)}(a^{(2)})^2\right)+\frac{1}{2}m_0^{(1)}(a^{(1)})^2+\frac{1}{2}m_0^{(2)}(a^{(2)})^2\\&-\frac{1}{2}\sum_{f=1}^{4}(m_f^{(1)})^2-\frac{1}{2}\sum_{f=1}^{3}(m_f^{(2)})^2-a^{(1)}(m_{\rm bif})^2\\&+\frac{1}{6}\sum_{f=1}^{4}\sum_{s_1=\pm1}\left[\!\left[a^{(1)}+s_1m_f^{(1)}\right]\!\right]^3+\frac{1}{6}\sum_{f=1}^{3}\sum_{s_2=\pm1}\left[\!\left[a^{(2)}+s_2m_f^{(2)}\right]\!\right]^3\\&+\frac{1}{6}\sum_{s_1=\pm1}\sum_{s_2=\pm1}\left[\!\left[a^{(1)}+s_1a^{(2)}+s_2m_{\rm bif}\right]\!\right]^3\\&=F_{\rm IMS}^{[4]-SU(2)-SU(2)-[3]}\end{aligned}$$

Example of UV-duality (triality)

Complete Prepotential for Rank 2 SCFT with SO(20) global symmetry

$m_0^{Sp} \gg |a_i^{Sp}|, |m_f^{Sp}|$ → **IMS prepotential for Sp(2) + 9 flavor**

$m_0^{SU} \gg |a_i^{SU}|, |m_f^{SU}|$ → **IMS prepotential for $SU(3)_{1/2}$ + 9 flavor**

$m_0^{(1)}, m_0^{(2)} \gg |m_f^{(1)}|, |m_f^{(2)}|, |a^{(1)}|, |a^{(2)}|$ → **IMS prepotential for [4]—SU(2)—SU(2)—[3]**

Property of complete prepotential

0. Defined for the whole parameter region

1. Reproduce the IMS prepotential [Intriligator, Morrison, Seiberg '97]

- Agree with IMS Prepotential inside the parameter region for the gauge theory.
- Additional corrections may appear outside.

2. Consistent with the 5-brane web

$$\frac{\partial F}{\partial \phi_i} = (\text{Monopole tension}) = (\text{Area of D3-brane in the web})$$

3. Respect the global symmetry

Invariant under the Weyl group of the global symmetry of the SCFT.

4. Consistent with UV-duality

Reproduce two or more different IMS prepotentials depending on the parameter region.

5. Derived from the Nekrasov partition function.

Nekrasov partition function for 5D gauge theory on S¹

$$Z_{\mathbb{R}^4\times S^1}(e^{-Ra_i}, e^{-Rm_0}, e^{-Rm_f}, e^{-R\epsilon_{1,2}})$$

$$= \underline{Z^{\text{pert}}_{\mathbb{R}^4\times S^1}(e^{-Ra_i}, e^{-Rm_f}, e^{-R\epsilon_{1,2}})} \left(1 + \underline{\sum_{k=1}^{\infty} Z^{k\text{-inst}}_{\mathbb{R}^4\times S^1}(e^{-Ra_i}, e^{-Rm_f}, e^{-R\epsilon_{1,2}}) e^{-\frac{1}{2}kRm_0}}\right)$$

Perturbative part

Instanton part

Perturbative Prepotential for 5D gauge theory on S¹

$$F^{\text{pert}}_{\mathbb{R}^4\times S^1} = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \log Z^{\text{pert}}_{\mathbb{R}^4\times S^1}$$

IMS prepotential for 5D gauge theory

$$F_{\text{IMS}} = \lim_{R\to\infty} \frac{1}{R} F^{\text{pert}}_{\mathbb{R}^4\times S^1}$$

Instanton effect disappears if $m_0 \gg |m_f|, |a_i|$

Nekrasov partition function for 5D gauge theory on S¹

$$Z_{\mathbb{R}^4\times S^1}(e^{-Ra_i}, e^{-Rm_0}, e^{-Rm_f}, e^{-R\epsilon_{1,2}})$$

$$= \underbrace{Z^{\text{pert}}_{\mathbb{R}^4\times S^1}(e^{-Ra_i}, e^{-Rm_f}, e^{-R\epsilon_{1,2}})}_{\text{Perturbative part}} \left(1 + \underbrace{\sum_{k=1}^{\infty} Z^{k\text{-inst}}_{\mathbb{R}^4\times S^1}(e^{-Ra_i}, e^{-Rm_f}, e^{-R\epsilon_{1,2}}) e^{-\frac{1}{2}kRm_0}}_{\text{Instanton part}}\right)$$

Perturbative part

Instanton part

➡ Prepotential for 5D gauge theory on S¹

$$F_{\mathbb{R}^4\times S^1} = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \log Z_{\mathbb{R}^4\times S^1}$$

➡ Complete prepotential for 5D SCFT

$$F_{\text{Complete}} = \lim_{R\to\infty} \frac{1}{R} F_{\mathbb{R}^4\times S^1}$$

Additional terms appears from instanton part
if we consider the whole parameter region

Complete prepotential for 5D SCFT

$$F_{\text{Complete}} = F_{\text{IMS}} - \frac{1}{6} \sum_{\substack{n_f, n_i \\ n_0 > 0}} (-1)^{j_L + j_R + 1} (2j_L + 1)(2j_R + 1) N^{j_L, j_R}_{n_i, n_0, n_f} \llbracket Z_{n_i, n_0, n_f} \rrbracket^3$$

$$Z_{n_i, n_0, n_f} := \sum_i n_i a_i + n_0 m_0 + \sum_f n_f m_f$$

$N^{(j_L, j_R)}_{(n_i, n_0, n_f)}$: Gopakumar-Vafa invariant

Problem: Contribution from infinitely many BPS particles

Resolution: For almost all the BPS particles,

$$Z_{n_i, n_0, n_f} \geq 0 \quad \Rightarrow \quad \llbracket Z_{n_i, n_0, n_f} \rrbracket^3 = 0$$

inside the "Physical Coulomb moduli" $\left(\frac{\partial F_{\text{Complete}}}{\partial a_i} \geq 0 \right)$

[P. Jefferson, H. Kim, C. Vafa, G. Zafrir '17]

Summary of complete prepotential

0. Defined for the whole parameter region

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Reproduce two or more different IMS prepotentials depending on the parameter region.

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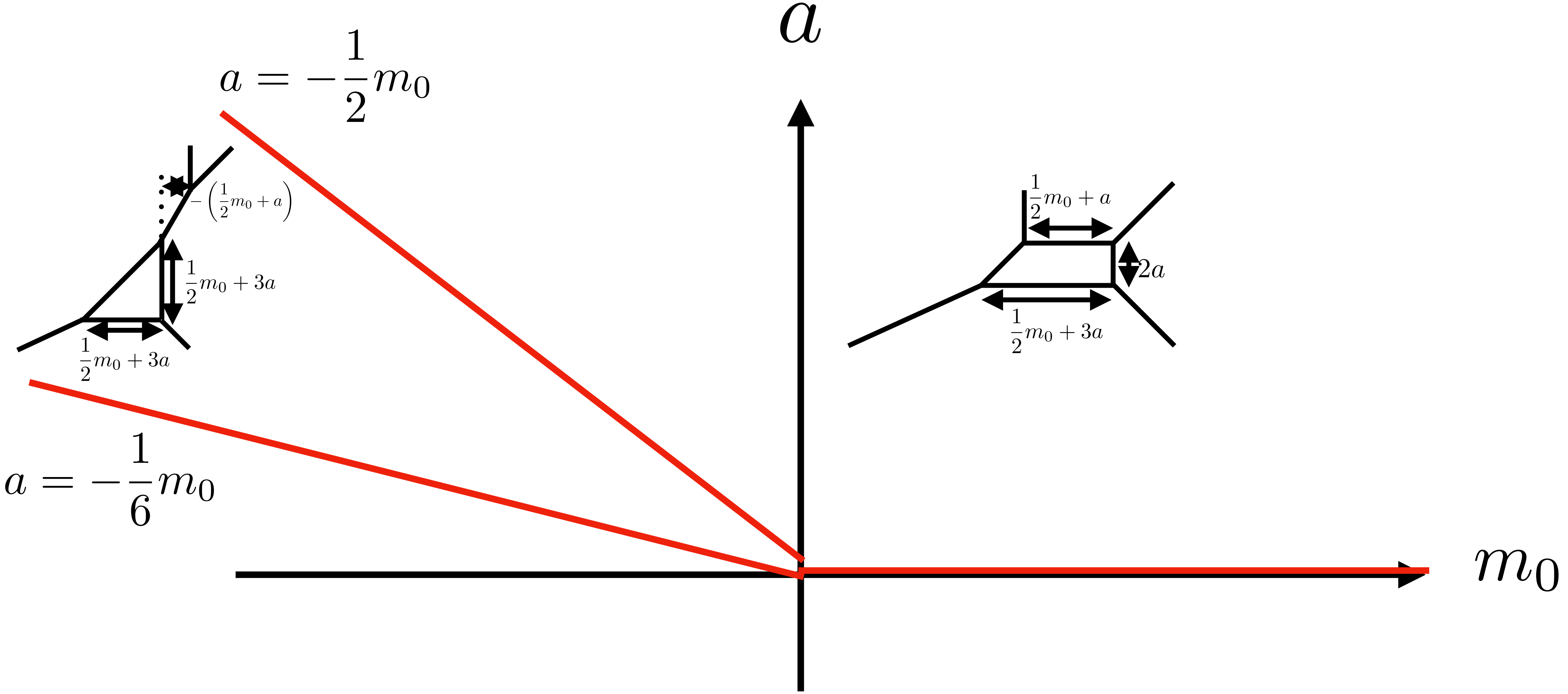
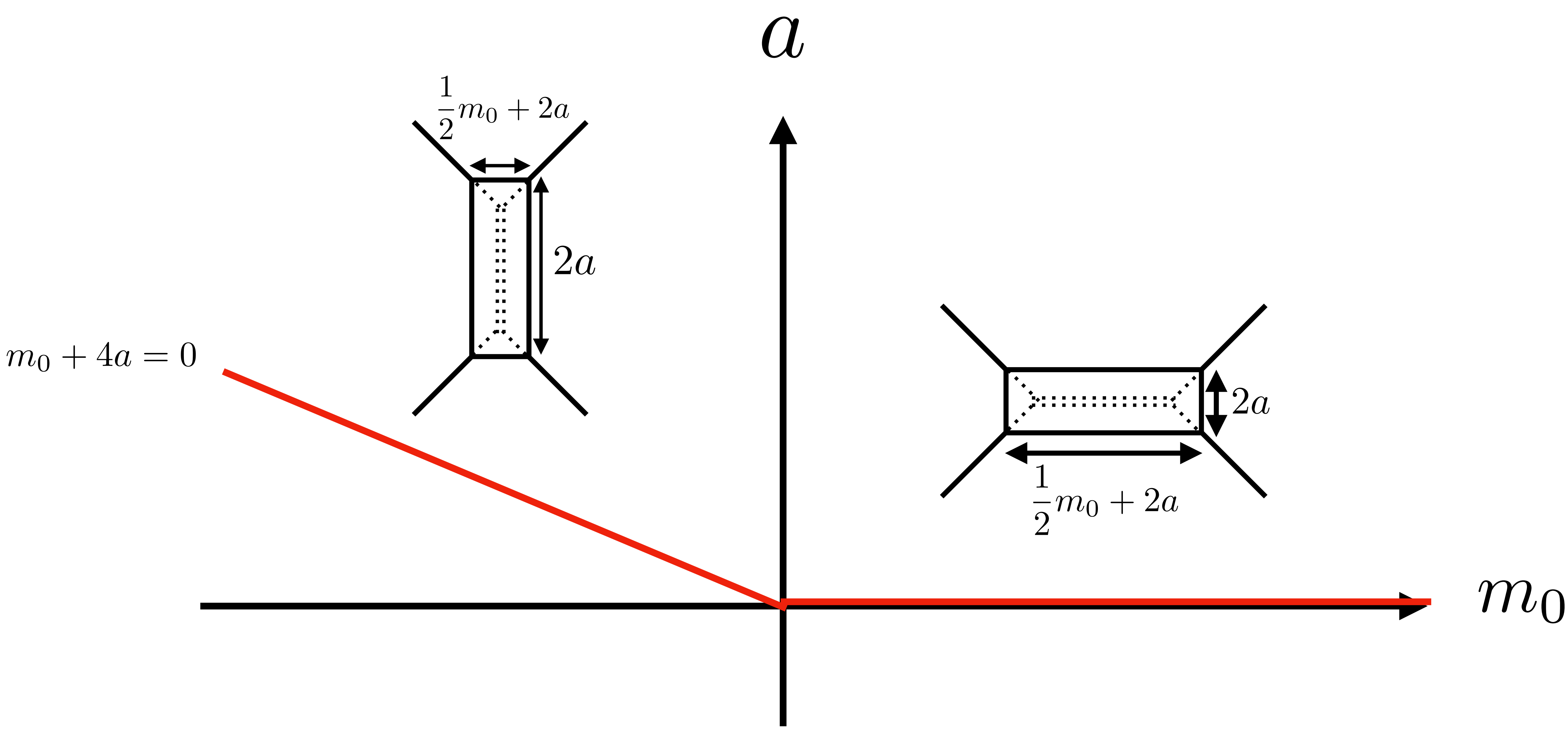
Thank you!

Parameter Space



Parameter Space



Key technique: Gopakumar-Vafa invariant

$$Z_{\mathbb{R}^4\times S^1} = Z_0 \exp\left(\sum_{(n_0,n_i,n_f)} \sum_{j_L,j_R} \sum_{n=1}^{\infty} \frac{N^{(j_L,j_R)}_{(n_i,n_0,n_f)} [j_L, j_R]_{t^n,q^n}}{n\left(t^{\frac{n}{2}} - t^{-\frac{n}{2}}\right)\left(q^{\frac{n}{2}} - q^{-\frac{n}{2}}\right)} e^{-nRT_{(n_i,n_0,n_f)}} \right)$$

$$t = e^{+R\epsilon_1}, \quad q = e^{-R\epsilon_2} \qquad [j_L, j_R]_{t,q} := (-1)^{j_L+j_R+1} \sum_{k=-j_L}^{j_L} (tq)^k \sum_{\ell=-j_R}^{j_R} (tq^{-1})^\ell$$

$$T_{(n_i,n_0,n_f)} := \sum_i n_i a_i + n_0 m_0 + \sum_f n_f m_f$$

$N^{(j_L,j_R)}_{(n_i,n_0,n_f)}$: Gopakumar-Vafa invariant