Refined topological strings on compact Calabi-Yau spaces

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Based on MH, S. Katz and A. Klemm, work in progress.

Background and Introduction

- Topological strings arise in compactification of superstring theory on Calabi-Yau 3-folds, computing certain effective actions in the 4D theory. Geometric engineering can relate physical (strong coupling) questions of 4-D quantum field theory to geometric questions of Calabi-Yau manifolds.

- Topological string theory is a tractable, computable sector of superstrings, and is an ideal setting to study fundamental ideas e.g. D-brane, S-duality, open/close string duality, etc.

- Mirror symmetry relates topological A-model on manifold X to topological B-model on its mirror manifold. Some very difficult mathematical problems of enumerative geometry, e.g. computing Gromov-Witten invariants, can be easily solved by topological B-model methods.

- Relations to matrix models, quantum integrable systems, black hole physics, etc.

Physical definition

- Consider maps from worldsheet, a genus g Riemann surface Σ_g, to a Calabi-Yau 3-fold M. We have a supersymmetric non-linear sigma model, apply topological twist (A-twist or B-twist). Integrate over the moduli space of the worldsheet.

- We are interested in the topological string partition function

$$Z = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}(t_i)\right)$$

where t_i are Kahler moduli in the case of A-model, and complex structure moduli in the case of B-model.

Gromov-Witten invariants

- Consider a Kahler manifold X and $\beta \in H_{(1,1)}(X, \mathbb{Z})$. Define moduli space of stable maps of genus g with n marked points, $\bar{\mathcal{M}}_{g,n}(X, \beta)$. Yong-Bin Ruan, Gang Tian, 1995; M. Kontsevich, 1995.

- The Gromov-Witten invariant is defined

$$N_g^{\beta} = \int_{\bar{\mathcal{M}}_{g,0}(X,\beta)} 1 \tag{1}$$

  The genus g amplitude is $F^{(g)}(t_i) = \sum_{\beta} N_g^{\beta} e^{-\beta \cdot t}$.

- The Gromov-Witten invariant is well defined if the virtual dimension of $\bar{\mathcal{M}}_{g,n}(X, \beta)$ vanishes. Two group of mathematicians later proved that the genus zero results are the same with those from mirror symmetry. A. Givental, 1996; B. Lian, K. Liu, and S.-T. Yau, Mirror principle I, II, III, 1997, 1999, 1999.

- Higher genus: Topological strings on non-compact toric Calabi-Yaus are essentially solved by topological vertex formalism.
M. Aganagic, A. Klemm, M. Mariño and C. Vafa, "The Topological vertex," [hep-th/0305132].
J. Li, C. C. M. Liu, K. Liu and J. Zhou, "A Mathematical theory of the topological vertex," [math/0408426 [math.AG]].

- A long standing problem: How to solve topological strings on compact Calabi-Yau spaces? At higher genus, the most effective approach is the mirror symmetry and use holomorphic anomaly equation M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa (BCOV), 1993. This was done by BCOV up to genus 2.

- The BCOV holomorphic anomaly equation

$$\bar{\partial}_{\bar{k}}\partial_m F^{(1)} = \frac{1}{2}\bar{C}_{\bar{k}}^{ij} C_{mij}^{(0)} + \left(\frac{\chi}{24} - 1\right) G_{\bar{k}m} \ ,$$

$$\bar{\partial}_{\bar{k}} F^{(g)} = \frac{1}{2}\bar{C}_{\bar{k}}^{ij} \left(D_i D_j F^{(g-1)} + \sum_{r=1}^{g-1} D_i F^{(r)} D_j F^{(g-r)} \right) \ , \quad g \geq 2$$

- We made some important progress Huang, Klemm, Quackenbush, hep-th/0612125.

  1. We discover boundary conditions at the conifold point of the moduli space, i.e. the "gap" condition c.f. Huang, Klemm, hep-th/0605195, which fix the holomorphic ambiguity to a large extend.

  2. We solve the holomorphic anomaly equation directly without the BCOV Feynman diagrams, by using the idea of formulating topological strings as polynomials Yamaguchi, Yau, hep-th/0406078. The computational complexity of the method grows only polynomially in genus.

- We are able to solve a class of one-parameter Calabi-Yau models to very high genus, e.g. in principle to genus 51 for the quintic.

Elliptic Calabi-Yau threefolds

- Elliptic Calabi-Yau threefolds were studied in the early days of mirror symmetry. Some recent works study the polynomial formalism and modular anomaly equations.
  M. Alim and E. Scheidegger, arXiv:1205.1784 [hep-th];
  A. Klemm, J. Manschot and T. Wotschke, arXiv:1205.1795 [hep-th].

- Some recent developments on the elliptic genus of E-strings, which is related to the topological strings on half K3 space, a local limit of elliptic fibration over $\mathbb{F}_1$.
  Haghighat, Lockhart and Vafa, arXiv:1406.0850 [hep-th];
  Cai, MH and Sun, arXiv:1411.2801 [hep-th];
  J. Kim, S. Kim, Lee, Park and Vafa, arXiv:1411.2324 [hep-th];
  Haghighat, Klemm, Lockhart and Vafa, arXiv:1412.3152 [hep-th].

- Based on these earlier works, using various conditions, notably the involution symmetry, and the weak Jacobi forms, we make some more progress toward solving the compact models.

The use of weak Jacobi forms

- Consider a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ depend on a modular parameter $\tau \in \mathbb{H}$, an elliptic parameter $z \in \mathbb{C}$. They transform under the modular group as

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \varphi(\tau, z), \qquad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}(2;\mathbb{Z})$$

  and under translations of the elliptic parameter as

$$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)} \varphi(\tau, z), \quad \forall \quad \lambda, \mu \in \mathbb{Z} \ .$$

  Here $k \in \mathbb{Z}$ is called the *weight* and $m \in \mathbb{Z}_{>0}$ is called the *index*.

- Due to the periodicity, the function has a Fourier expansion

$$\phi(\tau, z) = \sum_{n,r} c(n,r) q^n y^r, \qquad \text{where} \quad q = e^{2\pi i\tau}, \quad y = e^{2\pi i z}$$

- A (holomorphic) Jacobi form: $c(n,r) = 0$ unless $4mn \geq r^2$.
  (stronger) A cusp Jacobi form: $c(n,r) = 0$ unless $4mn > r^2$.
  (weaker) A weak Jacobi form : $c(n,r) = 0$ unless $n \geq 0$.

- Some weak Jacobi forms can be constructed by theta functions

$$\begin{aligned} \phi_{-2,1}(\tau,z) &= -\frac{\theta_1(z,\tau)^2}{\eta^6(\tau)}, \\ \phi_{0,1}(\tau,z) &= 4[\frac{\theta_2(z,\tau)^2}{\theta_2(0,\tau)^2} + \frac{\theta_3(z,\tau)^2}{\theta_3(0,\tau)^2} + \frac{\theta_4(z,\tau)^2}{\theta_4(0,\tau)^2}]. \end{aligned}$$

- M. Eichler and D. Zagier, The theory of Jacobi forms, 1985.
  The graded ring of (holomorphic) Jacobi forms with fixed index is a free module over modular forms.
  The bigraded ring of (holomorphic) Jacobi forms is not finitely generated.
  A weak Jacobi form of given index m and even modular weight k is a polynomial of $E_4(\tau)$, $E_6(\tau)$, $\phi_{0,1}(\tau,z)$, $\phi_{-2,1}(\tau,z)$ whose modular weights and indices are $4, 6, 0, -2$ and $0, 0, 1, 1$ respectively.

- The even weight weak Jacobi forms have a Taylor expansion in z with coefficients are quasi-modular forms. For example the first coefficients in the expansion of $\phi_{-2,1}(z,\tau)$ and $\phi_{0,1}(z,\tau)$ are

$$\begin{aligned}\phi_{-2,1}(\tau,z) &= -z^2+\frac{E_2z^4}{12}+\frac{-5E_2^2+E_4}{1440}z^6+\mathcal{O}(z^8),\\ \phi_{0,1}(\tau,z) &= 12-E_2z^2+\frac{E_2^2+E_4}{24}z^4+\mathcal{O}(z^6).\end{aligned}$$

- They satisfy the modular anomaly equation

$$\left(\partial_{E_2}+\frac{z^2}{12}\right)\phi_{-2,1}(\tau,z)=0,\qquad \left(\partial_{E_2}+\frac{z^2}{12}\right)\phi_{0,1}(\tau,z)=0.$$

- Therefore a weak Jacobi form $\varphi_{k,m}$ of index m satisfies the modular anomaly equation that

$$\left(\partial_{E_2}+\frac{mz^2}{12}\right)\varphi_{k,m}=0\ .$$

- We apply the idea to compact elliptic Calabi-Yau manifolds (without refinement). We expand the partition function on base degree

$$Z = Z_0[1 + \sum_{d_B=1}^{\infty} Z_{d_B}(\lambda, \tau) Q_B^{d_B} \frac{Q_E^{\frac{3d_B}{2}}}{\eta^{36 d_B}(\tau)}]$$

- We conjecture

$$Z_{d_B}(z, \tau) = \frac{\varphi_{d_B}(z, \tau)}{\prod_{k=1}^{d_B} \phi_{-2,1}(\tau, kz)},$$

where $z = \lambda$ is the genus expansion parameter of topological strings.

- According to the modular anomaly equation, $Z_{d_B}(z, \tau)$ has formally index $\frac{d_B(d_B-3)}{2}$. Modular anomaly equations in previous literature are automatically satisfied, with modular ambiguity fixed for all genus.

- Castelnuovo's bound: for a given degree (d_B, d_E), the GV invariant $n_g^{(d_B, d_E)}$ vanish at sufficiently large genus g. Only $\phi_{-2,1}(\tau, z)$ factor in the denominator in the remaining restricted ansatz.

- This approach can be combined with the B-model holomorphic anomaly approach to compute higher genus topological string amplitudes.
  Using the involution symmetry and boundary conditions, we find that the exact formula at base degree d_B can provide sufficient boundary data to fix the B-model formula at genus $9(d_B+1)$.
  On the other hand, in order to fix the exact (A-model) formula at base degree d_B, we need topological free energy of genus no less than $\frac{d_B(d_B-3)}{2}+1$.

- Thus, as long as $9(d_B+1) \geq \frac{(d_B+1)(d_B-2)}{2}+1$, we can repeat this procedure to fix the exact formula with increasing base degrees. In this way we can in principle determine the exact formula up to base degree $d_B = 20$ (for all genera and fiber degrees), and the topological string free energy up to genus 189 (for all base and fiber degrees). In practice we compute up to $d_B = 5$ and genus $g = 8$.

Refinements

- The Omega background was first introduced by Nekrasov, to regularize the integrals over moduli space of instantons in 4d $\mathcal{N} = 2$ Seiberg-Witten theory.
  The leading term of Nekrasov partition function is the prepotential in Seiberg-Witten theory, and we also have higher order terms from two expansion parameters $\epsilon_{1,2}$, which are interpreted as gravitational couplings.

- Motivated by the gauge theory calculations, one defines the refined topological string theory with two coupling constants $\epsilon_{1,2}$. Conventional topological string free energy of genus $g \geq 1$ on Calabi-Yau threefolds compute the effective actions term R^2F^{2g-2} in type IIA compactification to 4d. Now for refined topological string theory, we allow for both self-dual and anti-self-dual graviphoton field strength F and get a two-parameter expansion.

- The refined Gopakumar-Vafa invariants $n^{\beta}_{j_L,j_R}$ count 5d BPS particles with both $SU(2)$ spins from the decomposition of rotation group $SO(4)$, instead of an alternating sum over the right spin in the unrefined case. As a result, the unrefined Gopakumar-Vafa invariants may be negative integers but the refined Gopakumar-Vafa invariants are always non-negative.

- Mathematical definition of refined invariants: a refinement of Pandharipand Thomas's "stable pair invariants", making use of a C^* action on the on the stable pair moduli space. (J. Choi, S. Katz and A. Klemm, [arXiv:1210.4403]) The C^* action is not available in general compact Calabi-Yau manifolds, but the refinement may still be possible in elliptic Calabi-Yau spaces due to the elliptic fibration structure.

Refined holomorphic anomaly

- We can expand the free energy

$$F = \sum_{n,g=0}^{\infty} F^{(n,g)}(t_i)(\epsilon_1 + \epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}, \qquad (2)$$

where ϵ_1, ϵ_2 are gravi-photon field strength. The unrefined limit is $\epsilon_1 + \epsilon_2 = 0$, while the Nekrasov-Shatashvilli limit is $\epsilon_2 = 0$.

- The refined holomorphic anomaly equation is proposed. The results for non-compact toric geometry agree with the calculations from refined topological vertex.
D. Krefl and J. Walcher, [arXiv:1007.0263];
M. x. Huang and A. Klemm, [arXiv:1009.1126]

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(n,g)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}[D_j D_k \mathcal{F}^{(n,g-1)} + (\sum_{n_1=0}^{n}\sum_{g_1=0}^{g})' D_j \mathcal{F}^{(n_1,g_1)} D_k \mathcal{F}^{(n-n_1,g-g_1)}],$$

- 6d SCFTs can be constructed from F-theory on non-compact elliptic Calabi-Yau threefolds. Elliptic genus is computed by refined topological string theory.

- Simplest cases are known as the minimal 6d SCFTs. They have rank one in the tensor branch and realized as F-theory compactified on elliptic fibration over a non-compact base $\mathcal{O}(-n) \to \mathbb{P}^1$, where $n = 1, 2, \cdots 8, 12$.
  $n = 1, 2$: E-strings, M-strings, no gauge symmetry.
  $n > 2$: some gauge symmetries.

- We can also make a weak Jacobi form ansatz for the refined theory.
  M. Del Zotto and G. Lockhart, arXiv:1609.00310 [hep-th].
  J. Gu, MH, A. K. Kashani-Poor and A. Klemm, arXiv:1701.00764 [hep-th].
  M. Del Zotto, J. Gu, MH, A. K. Kashani-Poor, A. Klemm and G. Lockhart, arXiv:1712.07017 [hep-th].

- The ansatz for theories without gauge symmetry is

$$Z_\beta = \left(\frac{\sqrt{q}}{\eta(\tau)^{12}}\right)^{-\beta\cdot K_B} \frac{\phi_{k,n_+,n_-,\beta}(\tau,\boldsymbol{m},\epsilon_+,\epsilon_-)}{\prod_{i=1}^{r}\prod_{s=1}^{\beta_i}\left[\phi_{-1,\frac{1}{2}}(\tau,s\epsilon_1)\phi_{-1,\frac{1}{2}}(\tau,s\epsilon_2)\right]},$$

  Multiple sets of elliptic parameters: We can use weak Jacobi forms for each set independently.

- The geometric vanishing conditions completely fix the ansatz (up to a normalization), without the help of B-model.

- Generalization to the cases with gauge symmetry: more complicated ansatz. Use Weyl invariant Jacobi forms. Not completely fixed by *generic* vanishing conditions.

Refining the compact models

- There are some discussions that the (A-model) refined topological string theory on compact Calabi-Yau threefolds may not be consistent, as the amplitudes may depends on complex structure moduli.

- Some simple geometric calculations of the refined invariants are still available. With these data, we try to make sense of the refined theory with partial success. Our results may be only valid on a certain patch of the complex structure moduli space.

- The ansatz with weak Jacobi forms does not seem to work for refined theory on compact models (more later). Here we use the B-model approach of refined holomorphic anomaly equations.

Geometric Calculations

- A rigorous mathematical construction of Gopakumar-Vafa invariants base on "Perverse Sheaves".
  Y.-H. Kiem and J. Li, arXiv:1212.6444.
  Later, A problem with the construction was identified. But there is no problem in our simple situation, where the moduli spaces are smooth.
  D. Maulik and Y. Toda, Gopakumar-Vafa invariants via vanishing cycles, arXiv:/1610.07303[MATH.AG].

- We will use the Lefschetz decomposition of the Hodge cohomology into $SU(2)$ representations. Denote M the elliptic Calabi-Yau 3-folds over the base B, and their Lefschetz representations R_M, R_B.
  For $B = \mathbb{P}^2$, $h^{1,1}(M) = 2, h^{1,2}(M) = 272$, we have

$$R_B = [1]\,,\ R_M = \left[\frac{3}{2}\right] + \left[\frac{1}{2}\right] + 546\,[0]\,, \tag{3}$$

  For $B = \mathbb{F}_0$ or $B = \mathbb{F}_1$, $h^{1,1}(M) = 3, h^{1,2}(M) = 243$, we have

$$R_B = [1] \oplus [0]\,,\ R_M = \left[\frac{3}{2}\right] + 2\left[\frac{1}{2}\right] + 488\,[0]\,. \tag{4}$$

Base Degree 0

- The moduli spaces of stable sheaves $\mathcal{F}$ on M with Euler characteristic $\chi(\mathcal{F}) = -1$ is $M_{-1}((0,d)) \simeq M$ for any fiber degree d. The refined Gopakumar-Vafa invariants are the same for all d.

$$\sum_{j_L, j_R} N^{(0,d)}_{j_L, j_R}[j_L, j_R] = \left[\frac{1}{2}, R_B\right] \oplus \left[0, R_M - \left(\left[\frac{1}{2}\right] \otimes R_B\right)\right]. \tag{5}$$

- Use the previous equations, we compute
For $B = \mathbb{P}^2$, we have

$$\sum_{j_L, j_R} N^{(0,d)}_{j_L, j_R}[j_L, j_R] = \left[\frac{1}{2}, 1\right] \oplus 546\,[0,0]\,, \tag{6}$$

For $B = \mathbb{F}_0$ or $B = \mathbb{F}_1$, we have

$$\sum_{j_L, j_R} N^{(0,d)}_{j_L, j_R}[j_L, j_R] = \left[\frac{1}{2}, 1\right] \oplus \left[\frac{1}{2}, 0\right] \oplus 488\,[0,0]\,. \tag{7}$$

A salient feature: the breaking of "chess board" pattern (non-vanishing refined GV invariants have definite sign $(-1)^{2(j_L+j_R)}$) appearing for non-compact models.

- The fact that the refined Gopakumar-Vafa invariants are the same for all fiber degrees fits nicely with modularity at zero base degree. The single cover contribution to A-model free energy is

$$F_{single} \sim \sum_{d_E=1}^{\infty} q^{d_E} \sum_{n,g} (\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1} F_{single}^{(n,g)}, \tag{8}$$

where q is the exponential fiber Kahler parameter.

- To include the multi-cover contributions, we shall divide the above single cover contribution by a natural number m, rescale the parameters $q \to q^m, \epsilon_{1,2} \to m\epsilon_{1,2}$, and sum over m. We recognize the well-known formula for Eisenstein series $E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{m,d=1}^{\infty} m^{2k-1} q^{md}$. So we have

$$F_0^{(n,g)} \sim E_{2n+2g-2}(q), \quad n+g \geq 2 \tag{9}$$

with an appropriate chosen "refined constant map contributions", in terms of Bernoulli numbers.
Special treatments for $n+g<2$.

Some non-zero Base Degrees

- Fiber degree $d_E = 0$ is simply the local model. Here we consider $d_E = 1$.

- An example: $B = \mathbb{P}^2$, $d_B = 1$. The lines in $\mathbb{P}^2$ containing a fixed $p \in \mathbb{P}^2$ are parametrized by $\mathbb{P}^1$, with Lefschetz [1/2], and the Lefschetz of B is [1]. So the Lefschetz of Chow(β) and the Lefschetz for $M_{-1}(\beta)$ are

$$R_{\mathrm{Chow}(\beta)} = [1]\left[\frac{1}{2}\right] = \left[\frac{3}{2}\right] \oplus \left[\frac{1}{2}\right],$$
$$R_{M_{-1}(\beta)} = \left[\frac{1}{2}\right]\left(\left[\frac{3}{2}\right] \oplus \left[\frac{1}{2}\right] \oplus 546\,[0]\right) = [2] \oplus 2\,[1] \oplus [0] \oplus 546\left[\frac{1}{2}\right]$$

Similar to equation (5), we have

$$\begin{aligned}\sum_{j_L,j_R} N^{(1,1)}_{j_L,j_R}[j_L,j_R] &= \left[\frac{1}{2}, R_{\mathrm{Chow}(\beta)}\right] \oplus \left[0, R_{M_{-1}(\beta)} - \left(\left[\frac{1}{2}\right] \otimes R_{\mathrm{Chow}(\beta)}\right)\right] \\ &= \left[\frac{1}{2},\frac{3}{2}\right] \oplus \left[\frac{1}{2},\frac{1}{2}\right] \oplus 546\left[0,\frac{1}{2}\right]. \qquad (10)\end{aligned}$$

- Other examples of refined invariants $\sum_{j_L,j_R} N^{\beta}_{j_L,j_R}[j_L, j_R]$ are computed by the geometric method. To summarize:

- For $B = \mathbb{P}^2$ we have

$$
\begin{aligned}
\beta = (d_B, d_E) = (0, d_E): &\qquad [\frac{1}{2}, 1] + 546[0, 0],\\
\beta = (d_B, d_E) = (1, 1): &\qquad [\frac{1}{2}, \frac{3}{2}] + [\frac{1}{2}, \frac{1}{2}] + 546[0, \frac{1}{2}],\\
\beta = (d_B, d_E) = (2, 1): &\qquad [\frac{1}{2}, 3] + [\frac{1}{2}, 2] + [\frac{1}{2}, 1] + 546[0, 2].
\end{aligned}
$$

- More cases

$$
\begin{aligned}
B = \mathbb{F}_1, \mathbb{F}_0: \quad &\beta = (d_E, d_2, d_3) = (d_E, 0, 0): \ [\frac{1}{2}, 1] + [\frac{1}{2}, 0] + 488[0, 0],\\
&\beta = (d_E, d_2, d_3) = (1, 0, 1): \ [\frac{1}{2}, 1] \oplus [\frac{1}{2}, 0] \oplus 488[0, 0],\\
B = \mathbb{F}_0: \quad &\beta = (d_E, d_2, d_3) = (1, 1, 1): \ [\frac{1}{2}, 2] \oplus 2[\frac{1}{2}, 1] \oplus [\frac{1}{2}, 0] \oplus 488[0, 1],
\end{aligned}
$$

where d_3 denoted the degree of fiber of the Hirzebruch surface. The case $d_2 \neq 0, d_3 = 0$ in elliptic $\mathbb{F}_1$ model reduces to half K3 model (E-strings), and is solved in previous works.

- To compare with B-model, we convert into the refined invariants $n^{\beta}_{g_L,g_R}$ in "integer basis".

$$\sum_{g_L,g_R} n^{\beta}_{g_L,g_R}[I^{g_L}, I^{g_R}] = \sum_{j_L,j_R} N^{\beta}_{j_L,j_R}[j_L, j_R], \qquad (11)$$

where $I = [\frac{1}{2}] + 2[0]$ and the sum g_L, g_R is over non-negative integers while the sum over j_L, j_R are over non-negative half-integers.

- Some salient features:
  Only a finite number of non-vanishing invariants for a fixed Kahler class β. Much more non-vanishing invariants, could be negative in the "integer basis".
  For geometric method, it is easier to compute high/top spin invariants, while the B-model computes recursively from low spins in "integer basis".
  Low spin invariants in one basis receive contributions from all higher spin invariants in the other basis.
  Unrefined invariants $n^{\beta}_{g} = n^{\beta}_{g,0}$, $N^{\beta}_{j_L} = \sum_{j_R} N^{\beta}_{j_L,j_R}(-1)^{2j_R+1}$.

- The refined GV invariants $n_\beta^{g_L,g_R}$ for $B=\mathbb{P}^2$ in the integer basis.
  $\beta=(d_B,d_E)=(0,d_E)$:

$g_L \backslash g_R$	0	1	2
0	540	8	-2
1	3	-4	1

  $\beta=(d_B,d_E)=(1,1)$:

$g_L \backslash g_R$	0	1	2	3
0	-1080	524	12	-2
1	-6	11	-6	1

  $\beta=(d_B,d_E)=(2,1)$:

$g_L \backslash g_R$	0	1	2	3	4	5	6
0	2700	-10760	11170	-4112	434	24	-2
1	15	-80	148	-128	56	-12	1

- Green numbers: $g_R=0$, unrefined invariants.

Refined genus one formula

- The unrefined genus one formula is well known, related to Ray-Singer torsion,

$$\begin{aligned}\mathcal{F}^{(0,1)} &= \frac{1}{2}(3+h^{1,1}-\frac{\chi}{12})K+\frac{1}{2}\log\det G^{-1} \\ &-\frac{1}{12}\log(\prod_k \Delta_k)-\frac{1}{24}\sum_{i=1}^{h^{1,1}} s_i\log(z_i), \end{aligned} \quad (12)$$

  where the numbers s_i are related to the second Chern class. For our model there are two conifold divisors Δ_k.

- For refined genus one amplitude, we propose

$$\mathcal{F}^{(1,0)}=\frac{1}{24}[\log(\prod_k \Delta_k)-\sum_{i=1}^{h^{1,1}} c_i\log(z_i)]+c_0K. \quad (13)$$

  This is similar to non-compact models studied before: no holomorphic anomaly, universal logarithmic cut at conifold. For compact models there is an extra piece from Kahler potential K.

- For elliptic $\mathbb{P}^2$ model, the modularity condition at zero base degree $\mathcal{F}^{(1,0)}|_{q_B\to 0} \sim \log(\eta(q_E))$ fixes the all constants except one.
  The involution symmetry that exchanges the two conifold divisors is a weaker condition.

- We use one further data point from geometric conditions to fix the remaining constant. The formula for elliptic $\mathbb{P}^2$ model

$$\mathcal{F}^{(1,0)} = \frac{1}{24}[\log(\Delta_1\Delta_2) + \frac{95}{2}\log(z_1) + \frac{89}{6}\log(z_2)] - \frac{101}{8}K.$$

The refined GV invariants $n^{g_L,g_R}_{(d_B,d_E)}$ for the genus $(g_L, g_R) = (0,1)$

$d_B\backslash d_E$	0	1	2	3	4
0		8	8	8	8
1	-4	524	4812	352294248	59107602110
2	35	-10760	1416596	-95518872	-93784873094
3	-386	193604	-43121628	5704148756	-636789454340
4	5161	-3477472	1076763820	-205276478472	32984800267144

Green numbers: $d_E = 0$ reduces to local model in previous works.
Two checks for the numbers in blue (less one for fixing formula).
See previous page for geometric calculations.
Geometric meaning for the coefficients?

- For elliptic $\mathbb{F}_1$ model, the fiber modularity and E-string results fix the refined genus one formula up to a constant. We have one non-trivial check from geometric calculations (numbers in blue, less one).
  The refined GV invariants $n^{g_L,g_R}_{(d_E,d_2,d_3)}$ for the genus $(g_L, g_R) = (0, 1)$

$(d_2, d_3)\backslash d_E$	0	1	2	3
(0,0)		8	8	8
(0,1)	1	8	128176	15504320
(1,0)	0	-2	-510	-11780
(0,2)	0	0	8	15504320
(1,1)	-4	464	3762	352886448
(2,0)	0	0	5890	651720

- For elliptic $\mathbb{F}_0$ model, the fiber modularity and the symmetry in the base fix the refined genus one formula up to a constant. We have two non-trivial checks from geometric calculations (numbers in blue, less one).
  The refined GV invariants $n^{g_L,g_R}_{(d_E,d_2,d_3)}$ for the genus $(g_L, g_R) = (0, 1)$.

$(d_2, d_3)\backslash d_E$	0	1	2	3
(0,0)		8	8	8
(0,1)	1	8	172836	19077120
(0,2)	0	0	8	19077120
(1,1)	10	-1896	104496	39263136

- Green numbers: local models. Yellow numbers: E-strings.

Higher genus

- For higher genus, we apply the refined holomorphic anomaly equation and refined boundary conditions at conifold divisors. We consider only elliptic $\mathbb{P}^2$ model. (The holomorphic anomaly equation for the BCOV propagators in elliptic $\mathbb{F}_1$ and $\mathbb{F}_0$ are difficult to solve.)

- The involution symmetry works as $\mathcal{F}^{(n,g)} \to (-1)^{n+g-1}\mathcal{F}^{(n,g)}$.

- The approach works for $n \leq 1$ (The case of $n = 0$ is unrefined). However for $n > 1$ the approach produces inconsistent fractional Gopakumar-Vafa invariants. (Our notation is a bit confusing. To compute refined GV invariants n^{g_L,g_R}, we need the topological amplitudes $\mathcal{F}^{(n,g)}$ for all $n \leq g_R, n+g \leq g_L + g_R$.)

- The result for $\mathcal{F}^{(1,1)}$ confirms 3 non-vanishing numbers (in red color, see previous pages for geometric calculations) from geometric calculations.

  The refined GV invariants $n^{g_L,g_R}_{(d_B,d_E)}$ for the genus $(g_L, g_R) = (1, 1)$.

$d_B \backslash d_E$	0	1	2	3
0		-4	-4	-4
1	0	11	-2077	120998
2	0	-80	29748	-4403178
3	165	-63523	11346927	-1225197859
4	-7448	4228100	-1121213060	185100882210

- We compute $\mathcal{F}^{(1,g)}$ up to $g = 3$, confirming the vanishings of $n^{g_L,g_R}_{(d_B,d_E)}$ for $g_R = 1, g_L = 2, 3$ and $(d_B, d_E) = (0, d_E), (1, 1), (2, 1)$.

- We can derive a modular anomaly equation from refined holomorphic anomaly equation. For the example of elliptic $\mathbb{P}^2$ model we have

$$\partial_{E_2} P_k^{(n,g)} = -\frac{1}{24}\sum_{s=1}^{k-1}\sum_{n_1=0}^{n}\sum_{g_1=0}^{g} s(k-s)P_s^{(n_1,g_1)}P_{k-s}^{(n-n_1,g-g_1)} + \frac{k(3-k)}{24}P_k^{(n,g-1)} - \frac{89k}{1728}P_k^{(n-1,g)}, \quad (14)$$

  where $P_k^{(n,g)}$ is the A-model genus (n,g) amplitude on base degree $k>0$. This is valid at least for $n=0,1$. The strange number $\frac{89}{1728}$ comes from the refined genus one amplitude $\mathcal{F}^{(1,0)}$.

- Weak Jacobi forms for refined theory on compact elliptic $\mathbb{P}^2$ model? Following similar derivations in E-string and M-string theories, we find the indices for $\epsilon_\pm = (\epsilon_1 \pm \epsilon_2)/2$ of the numerator in the ansatz as

$$n_+\epsilon_+^2 + n_-\epsilon_-^2 = \frac{k(k+1)(2k+1)}{12}(\epsilon_1^2+\epsilon_2^2) + \frac{k(3-k)}{2}\epsilon_1\epsilon_2 - \frac{89k}{144}(\epsilon_1+\epsilon_2)^2. \quad (15)$$

  For example for $k=1$ we have $n_-=0, n_+=-\frac{17}{36}$, seems clearly inconsistent.

Summary and Conclusion

- We shall discover more structures in the topological string partition function, enable the complete solution on a compact Calabi-Yau three-fold.

- The refined holomorphic anomaly equation needs correction in the compact case for $n \geq 2$.

- We use geometric method and refined holomorphic anomaly method for computations. The results from two approaches have only a small overlap that we can check. Each approach makes predictions that are beyond the current scope of the other approach.

Thank You