HOLOGRAPHIC SUBREGION COMPLEXITY OF A 1+1 DIMENSIONAL *P*-WAVE SUPERCONDUCTOR

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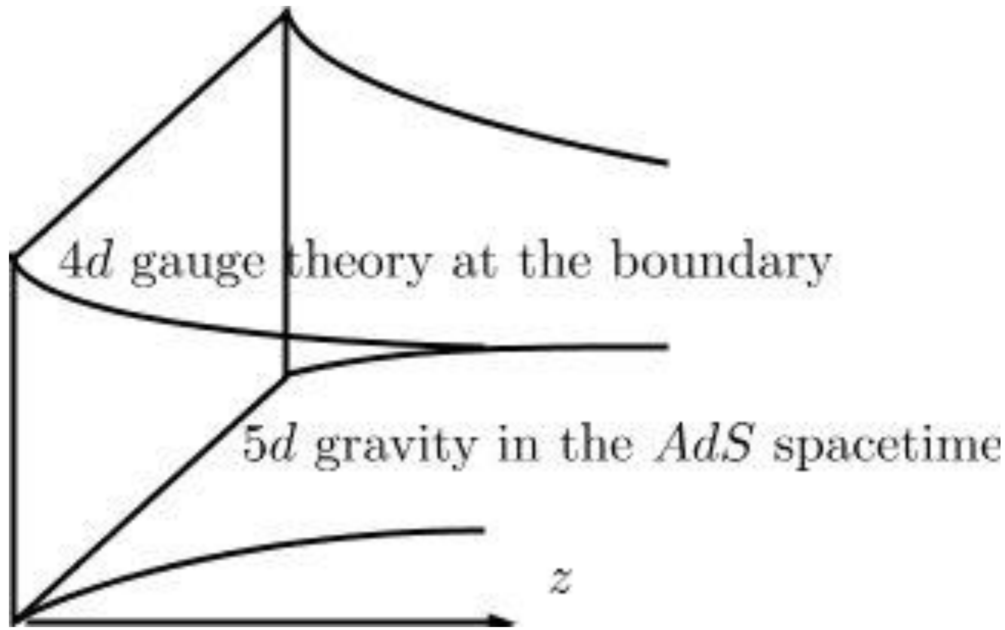


THE GAUGE/GRAVITY CORRESPONDENCE

d=4 N=4 SYM ⇔ IIB supergravity on $AdS_5 \times S^5$

Maldacena `97

Strongly coupled d=4 QFT ⇔ Weakly coupled d=5 gravity

The gauge/gravity correspondence

CFT	AdS
• Entanglement entropy	• A minimal surface (Ryu-Takayanagi formula `06)
• Complexity	• ?

The entanglement entropy (an extension of the thermal entropy)

- System whose total Hilbert space is a direct product:

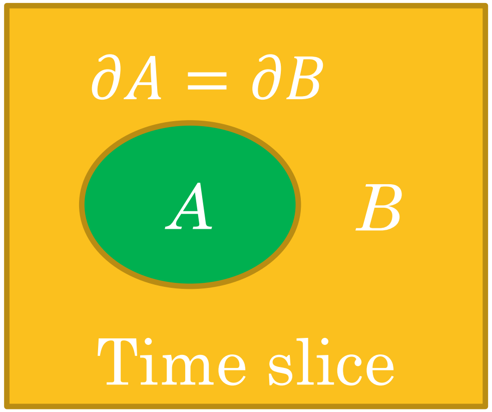
$$H = H_A \otimes H_B$$

- Definition of the reduced density matrix $\rho_A = Tr_B(\rho)$ taking the trace over H_B

- Entanglement Entropy (EE) defined using the density matrix ρ_A as

$$S_A = -Tr_A(\rho_A log \rho_A)$$

Von Neumann entropy of ρ_A



- In QFT, A and B: often a spatial bipartition of a system

The entanglement entropy is useful

- S^A measures how a given quantum state is entangled quantum mechanically

- An area law: $S^A \sim \gamma \frac{Area(\partial A)}{a^{d-1}}$ (a is lattice spacing)

  Bombelli-Koul-Lee-Sorkin `86, Srednicki `93

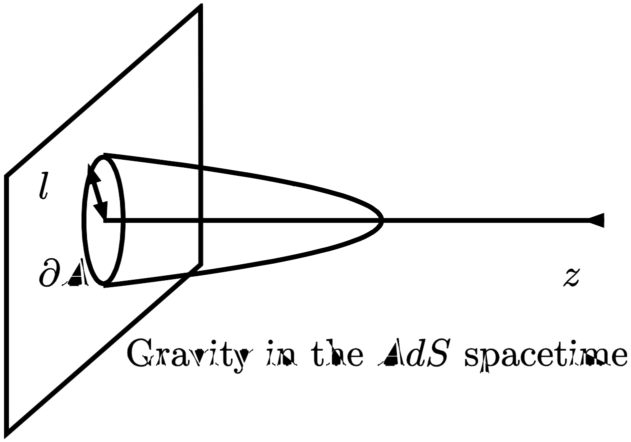
- Non-vanishing even at zero temperature
  - EE is proportional to degrees of freedom

- An order parameter (cf. Wilson loop)
  - It can capture the phase transition

The holography entanglement entropy (HEE)

- Holographic EE formula *Ryu-Takayanagi `06*

$$S^A = \frac{Area(\gamma_A)}{4G_N^{d+2}}$$



- The area of minimal surface γ_A (codimension=2) whose boundary is ∂A
- Like Bekenstein-Hawking formula when γ_A is the horizon
- γ_A is an extremal surface when the geometry is non-static

Hubeny-Rangamani-Takayanagi, `07

The gauge/gravity correspondence

QFT

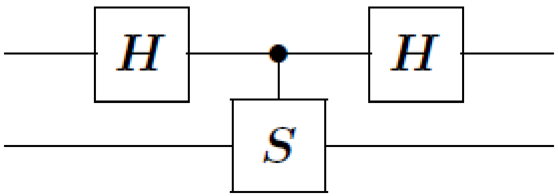
- Entanglement entropy
- Complexity

AdS

- A minimal surface (Ryu-Takayanagi formula `06)
- Maximal volumes or Action in the Wheeler-DeWitt patch

Complexity

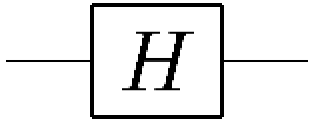
- Any quantum circuit: the minimum # of gates to obtain a desired target state from a reference state



$$|\psi> = U\,|\psi_0>$$

$|\psi_0> :$ a simple reference state

- U : unitary operator built from a set of simple gates (c.f. Hadamard gate acting on a single q-bit, S as a controlled phase gate)

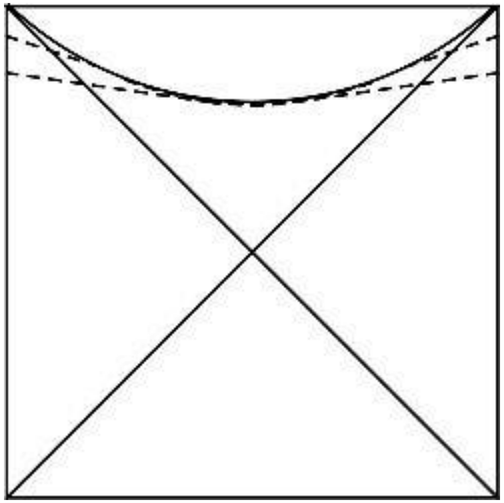


$$= H \otimes I \otimes I \otimes \ldots$$

Holographic complexity

- The holographic complexity = the surface *L(t)* of the Einstein-Rosen bridge *Susskind `14*

- The holographic complexity grows linearly in time as the length of the surface grows



From the complexity=volume to the complexity=action

- The complexity= the volume of the codimension-1 maximal bulk surface

$$C \sim \frac{V}{\kappa^2 l}\ , l = AdS\ radius, horizon\ radius\ \ldots$$

  - The coefficient becomes arbitrary

$$C \sim \frac{V l_{AdS}}{\kappa^2 l_{AdS}^2} \sim \frac{\Lambda}{\kappa^2}\int d^{d+1}x\sqrt{-g} \sim Action$$

- The complexity = the Einstein-Hilbert action in the Wheeler-DeWitt patch *Susskind `15*

The holographic complexity of the mixed state (subregion complexity)

- The holographic complexity by tracing out states of a separate region

- It is proportional to the volume surrounded by the extremal surface γ_A (Ryu-Takayanagi formula)

$$C = \frac{volume(\gamma_A)}{\kappa^2 l}$$

  - l is an arbitrary scale
  - Maximal volume attached to the extremal surface

Features of the subregion complexity

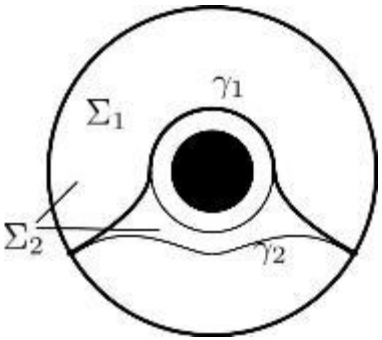
- The tensor network corresponding the *2d* Ising model:

$$C \propto \frac{l}{\epsilon}$$

Abt. et al `17

- An expected linear law behavior in AdS_3:

$$C \propto \frac{l}{\epsilon}$$



  - The finite part ⊃ χ (the Euler number)
  - A discontinuous jump at the transition : from Σ_1 to Σ_2

$$\Delta HC_u \sim 2\pi \Delta \chi$$

The motivation (the subregion complexity near a critical point)

- To probe a superconductor phase transition
  - Is HC_u(a universal part of C) finite?
  - Unlike the divergence $S \sim \frac{c}{6} \log \frac{\xi}{a}$ of a *2d* quantum critical phase transition *G. Vidal `02*
  - Finiteness of S across the holographic superconductor phase transition
  - Finite HC_u through the *1+1d* *s*-wave superconductor phase transition *Zangeneh-Ong-Wang `17*

- Doesn't the complexity behave like the entanglement entropy?

Topdown *2d* p-wave superconductor

- The brane configuration:

0	1	2	3	4	5	6	7	8	9
N D3	x	x	x						
N_FD 3	x			x	x				

- In the probe limit, D3-D3' realizes *SU(2)* Yang-Mills theory in an AdS_3 black hole *Gao-Kaminski-Zeng-Zhang `12、Yanyan `12*

- Holographic *2d p*-wave superconductor: the gap formation, the AC conductivity, and zero modes

- But, the dilaton would run with backreaction.

A toy model of *2d* p-wave superconductor

- The model: a *3d* gravity and *SU(2)* Yang-Mills theory with a negative cosmological constant

$$I_G = \frac{1}{2\kappa^2}\int d^3x\sqrt{-g}\left(R + \frac{2}{L^2} - \mathrm{Tr}(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})\right)$$

The field strength $\widetilde{F_{\mu\nu}} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - i g_{YM}[\tilde{A}_\mu, \tilde{A}_\nu]$

- Important aspect in *3d*: the gauge field in alternative quantization
  - gauge field describes the VEV of a gauge field living in the boundary
  - Dynamical gauge field $\longrightarrow$ superconductors as opposed to superfluids

An ansatz of the metric and fields

- An ansatz of the metric and fields

$$ds^2 = \frac{\tilde{L}^2}{z^2}\Big(-f_2(z)dt^2 + dy^2 + \frac{dz^2}{h_2(z)f_2(z)}\Big),$$
$$A = \frac{1}{2}\Big(\phi(z)\sigma^3 dt + w(z)\cdot\sigma^1 dy\Big), \quad A_z^b = 0,$$

  - σ^a $(a = 1,2,3)$: Pauli matrices and $f_2(z)$ is the blackening factor

- EOM:

$$D_\mu(\sqrt{-g}F^{\mu\nu}) = \partial_\mu(\sqrt{-g}F^{\mu\nu}) - i\sqrt{-g}[A_\mu, F^{\mu\nu}] = 0,$$
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Big(R + \frac{2}{\tilde{L}^2}\Big) = \kappa^2 T_{\mu\nu},$$

$$T_{\mu\nu} = \frac{2}{\kappa^2}\mathrm{tr}\Big(\tilde{F}_{\mu\alpha}\tilde{F}_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}\tilde{F}_{\alpha\beta}\tilde{F}^{\alpha\beta}\Big) = \frac{2}{\kappa^2 g_{YM}^2}\mathrm{tr}\Big(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big)$$

An ansatz of the metric and fields

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  - σ^a $(a = 1,2,3)$: Pauli matrices and $f_2(z)$ is the blackening factor

- EOM:

$$-\sqrt{h_2}f_2(z\sqrt{h_2}\phi')' + zw^2\phi = 0,$$
$$\sqrt{h_2}f_2(z\sqrt{h_2}f_2 w')' + z\phi^2 w = 0.$$

$$-\frac{zh_2 f_2' - 2f_2h_2 + 2}{z^2 f_2 h_2} - \frac{z^2\left(f_2h_2\left(f_2w'^2 - \phi'^2\right) + \phi^2 w^2\right)}{g_{YM}^2\tilde{L}^2 f_2^2 h_2} = 0,$$

$$\frac{f_2\left(zh_2f_2' + zf_2h_2' - 2f_2h_2 + 2\right)}{z^2} - \frac{z^2(\phi^2w^2 + f_2h_2(\phi'^2 + f_2w'^2))}{g_{YM}^2\tilde{L}^2} = 0,$$

$$\frac{2z^2h_2f_2'' + z^2f_2'h_2' - 4zh_2f_2' - 2zf_2h_2' + 4f_2h_2 - 4}{2z^2} + \frac{z^2\left(\phi^2w^2 - f_2h_2\left(f_2w'^2 + \phi'^2\right)\right)}{g_{YM}^2\tilde{L}^2 f_2} = 0,$$

  - EOM only depend on the combination $g_{YM}L$

The normal phase: a charged AdS_3 black hole solution ($w(z) = 0$)

- In the normal phase: a AdS_3 charged black hole on a circle
  $\Leftrightarrow$ The boundary theory: $T \neq 0, \ q \neq 0$
  - $A_\mu^{(3)}$ has the role of the usual Maxwell field

- The metric of AdS_3 charged black hole:

$$f_2(z) = 1 - \left(\frac{z}{z_0}\right)^2 + \frac{q^2}{g_{YM}^2} z^2 \log\left(\frac{z}{z_0}\right), \quad h_2(z) = 1, \quad \phi(z) = q \log\left(\frac{z}{z_0}\right)$$

$$T_H = \frac{1}{4\pi}\left(\frac{2}{z_0} - \frac{q^2 z_0}{g_{YM}^2}\right)$$

- The BPS-like bound: $M_0 = \left(\frac{L}{z_0}\right)^2 \geq \left(\frac{q}{\sqrt{2} g_{YM}}\right)^2$
  - The outer horizon z_0 is always smaller than z_{min} giving its extremal value $f(z_{min})<0$

The condensed phase: the black hole with a vector hair on a circle ($w(z) \neq 0$)

- The behavior of solutions near the *AdS* boundary (*z=0*)

$$
\begin{aligned}
\phi(z) &\sim q \log\left(\frac{z}{z_p}\right), \\
w(z) &\sim w_c + J_w \log(z), \\
f_2(z) &\sim \frac{1 - r_0^2 z^2}{h_0} + \frac{q^2}{g_{YM}^2} z^2 \log(z), \\
h_2(z) &\sim h_0,
\end{aligned}
$$

  - q: the charge density, w_c: VEV of a vector operator J_w : the source conjugate to w_c

- Satisfying superconducting boundary condition
  - A vanishing source $J_w=0$
  - A non-trivial solution signifying spontaneous symmetry breaking
    Breaking residual bulk *U(1)* generated by A_t^3 spontaneously
    c.f. $J_w \neq 0$ corresponds to the explicit breaking

Spontaneous symmetry breaking vs Explicit breaking in QFT

- **Spontanenous symmetry breaking ($J_w = 0, w_c \neq 0$)**
  : The symmetry is held in the action, but, it is broken by the ground state.

- **Explicit symmetry breaking ($J_w \neq 0$) : symmetry is broken in the EOM or the lagrangian. These symmetry breaking terms are used when these are small**

Cf. Chiral symmetry breaking in QCD: $SU(2)_L \times SU(2)_R \rightarrow SU(2)$

$\langle \overline{q_R}\, q_L \rangle = v\delta_{ab}$ (Spontaneous breaking)

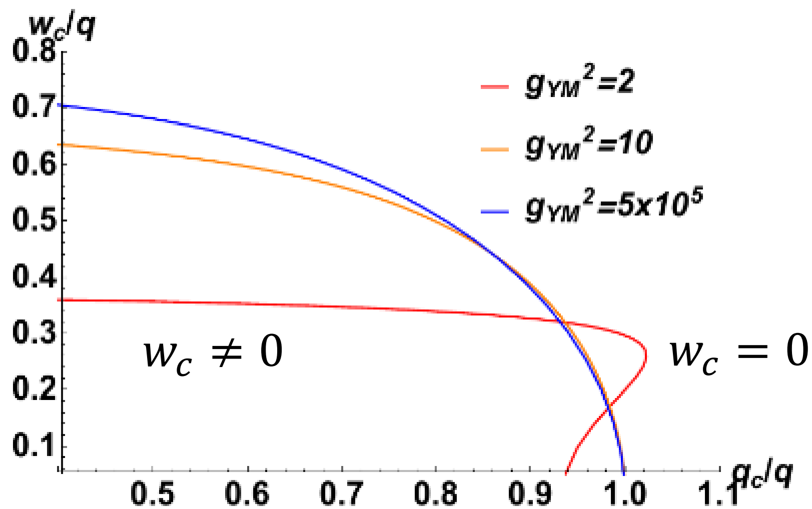
$L \supset \int d^4x\, \mathrm{m}\, \overline{q_R}\, q_L$+c.c. (Explicit breaking)

The expectation value of a vector operator

- The black hole with a vector hair: $A_x^{(1)} \neq 0 \;\; \Leftrightarrow$ a vector hair
  - The condensate (see Figure) of a vector operator: $A_x^{(1)}|_{z\to 0} \sim w_c$
  - Critical behavior around $q{=}q_c$

    $$\frac{w_c}{q} \sim 1.18\sqrt{1-\frac{q_c}{q}}$$

  - w_c suddenly jumps at a critical charge $q{=}q_c$ ($g_{YM}{}^2{<}3.2$)



Holographic entanglement entropy

- Ryu-Takayanagi formula $S^{EE} = \frac{2\pi Area(\gamma_A)}{\kappa^2}$
  - An interval on the boundary in the y direction of length l :

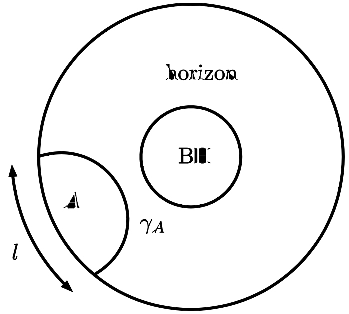
    $-\frac{l}{2} \leq y \leq \frac{l}{2}$

  - the EOM $z' = \sqrt{h(z)f(z)(\frac{z_*^2}{z^2} - 1)}$, z_* is the turning point ($z'=0$)

- Minimizing the action functional γ_A

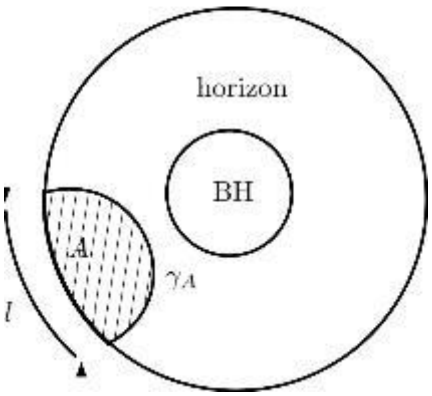
$$l_{curve} = 2\int_{z_*}^{\infty} dz \frac{1}{\sqrt{f_2(z)h_2(z)\left(\frac{z_*^2}{z^2} - 1\right)}}$$

$$S^{EE} = \frac{2\pi}{\kappa^2}(\gamma_A) = \frac{4\pi}{\kappa^2}\int_{\epsilon}^{z_*} dz \frac{1}{z\sqrt{f(z)h(z)(\frac{z^2}{z_*^2} - 1)}}$$



The holographic subregion complexity

- The holographic subregion complexity



$$C = \frac{\mathrm{volume}(\gamma_A)}{\kappa^2}$$

$$C(\epsilon) = \frac{c}{6\pi}\int_\epsilon^{z_*}\int_0^{x(z)} \frac{dz dx}{z^2\sqrt{f(z)h(z)}} = \frac{c}{6\pi}\int_\epsilon^{z_*} \frac{x(z)dz}{z^2\sqrt{f(z)h(z)}}$$

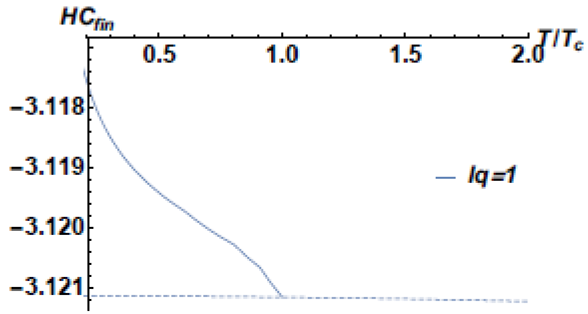
- The central charge $c = \frac{12\pi}{\kappa^2}$
- Scale invariant under $(t, y, z) \to \Lambda_0^{-1}(t, y, z)$
- The divergent structure $\sim l/\varepsilon$

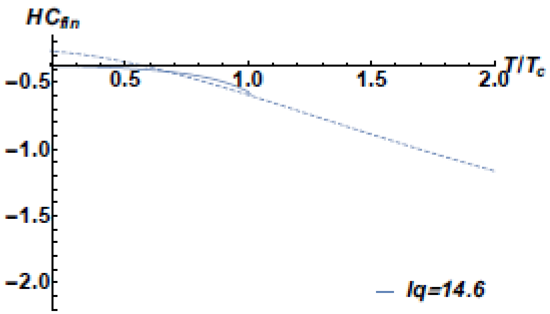
  (l is the size of A) *c.f. Carmi-Myers-Rath `16*

The holographic subregion complexity

- HC_u : the universal part of C

  HC_u as a function of T/T_c ($T_c = 0.03q = 0.3$ *and* $\kappa^2 = 0.1$)



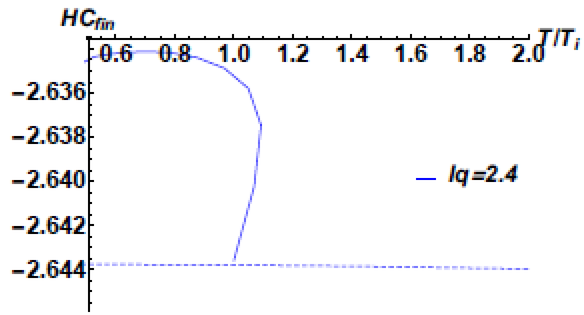


- HC_u decreases in both normal and condensed phases

  HC_u can be smaller in the condensed phase
- When *lq<<1*, two curves almost agree

The holographic subregion complexity

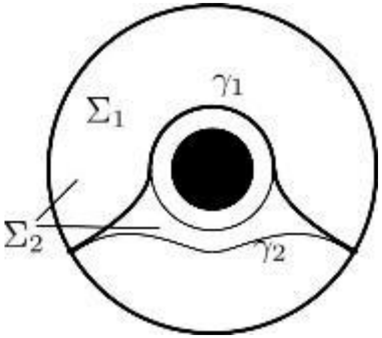
- $HC_u \quad (Tc = 0.0053q = 0.053 \; and \; \kappa^2 = 0.5)$



- HC_u decreases not in the condensed phase but in the normal phase with increase of T
- Multi-valued at a specific range of parameters

HC_u as a function of lq

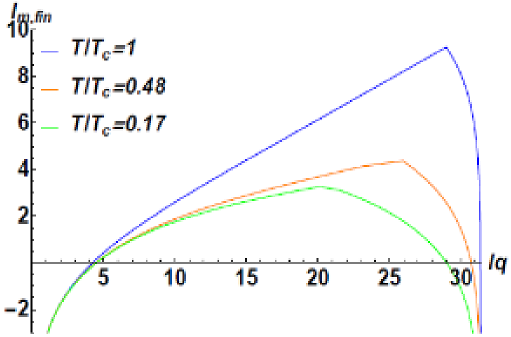
- $l < l_c$: S^{EE}=$S^{EE}(l)$ (γ_1)
  $l > l_c$: S^{EE}= (A) (horizon + γ_2)



Here A is

$$S^{EE} = S_{ent} + S^{EE}(2\pi - l)$$

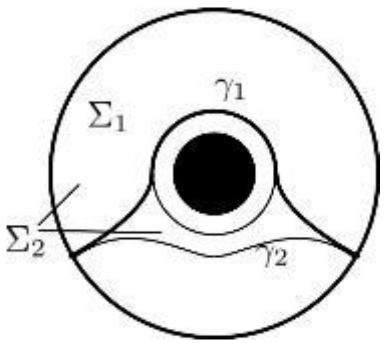
- S_{ent}: thermal entropy
- The finite part of the EE (q=5) decreases with decrease of temperature
  $l_c q$=29.1, 25.7, 20.5 for T/T_c=1, 0.48. 0.17



HC_u as a function of lq

- $l < l_c$: $C = C(l)$ (Σ_1)

  $l > l_c$: $C = C_{entire} - C(2\pi - l)$ (Σ_2)



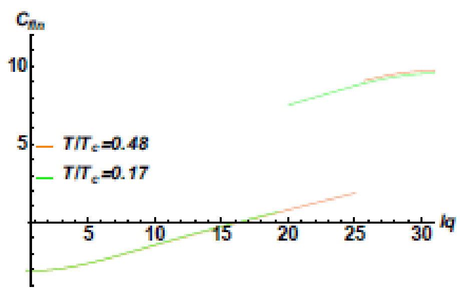
where the subregion complexity of the entire spatial boundary becomes

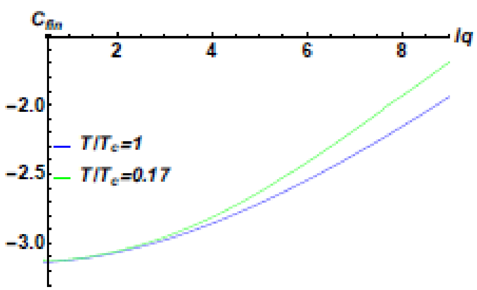
$$C_{entire} = \frac{c}{3}\int_{\epsilon}^{z_*} dz \frac{1}{z^2\sqrt{f(z)h(z)}}$$

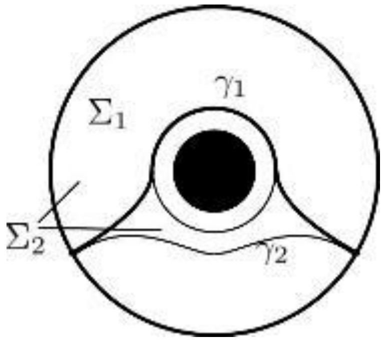
The critical size l_c does not depend on the magnitude of the subregion complexity

The discontinuous jump of *HC*

- The normalized finite part HC_u







- Sudden jump at a critical size $l_c q$=*29.1, 25.7, 20.5* for T/T_c=*1, 0.48. 0.17*
- The critical values decrease with decrease of T/Tc

Discussion

- The computation of the subregion complexity C by fixing q or T shows that HC_u is finite

- HC_u does not behave like the finite part of the holographic entanglement entropy

  - $\kappa^2 = 0.1 < 0.31$: HC_u decreases with increase of T/T_c or q_c/q (HC_u can be smaller in the condensed phase)
  - $\kappa^2 = 0.5 > 0.31$: non-monotonic behavior of HC_u
  - The phase transition does not agree with the one in the Ising model due to difference of models

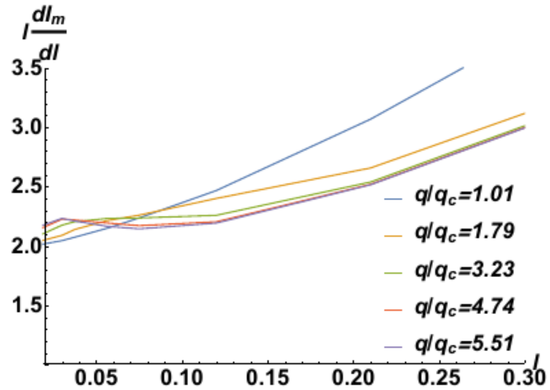
Thank you!

The renormalized entanglement entropy

- The *2d* renormalized EE in the holography $l\frac{\partial S^{EE}}{\partial l}$
(c.f. $S_{RE} = \frac{1}{(d-2)!!} R\frac{d}{dR}\left(R\frac{d}{dR} - 2\right)\cdots\left(R\frac{d}{dR} - (d-2)\right)S^{EE}$ in higher dimensions d even)
  - UV finite and no cutoff dependence
  - DOF at the scale of the size of the interval l
  - Small lT limit: the renormalized EE approaches to the vacuum behavior, which is a constant

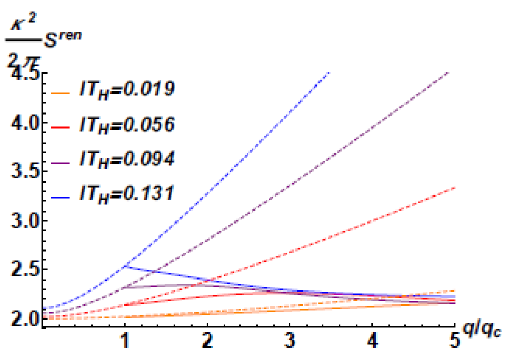
Renormalized EE in a holographic p-wave superconductor

- Renormalized EE as a function of l ($g_{YM}^2 = 10$)

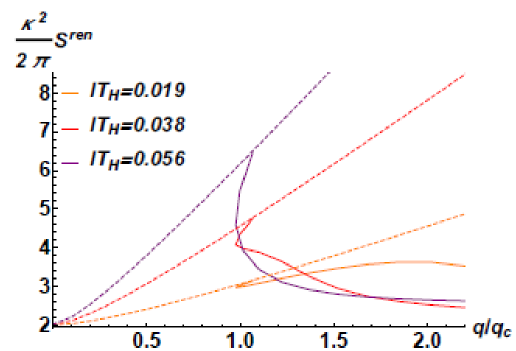


  - There appear extremal values.

- Renormalized EE as a function of q/q_c



($g_{YM}^2 = 10$)



($g_{YM}^2 = 2$)