Black holes in AdS/CFT

Seok Kim
(Seoul National University)

String theory and QFT, Fudan University
Mar 15, 2019

Statistical approaches to BH

- 5d BPS BHs from D1-D5-P. Cardy formula of 2d CFT [Strominger, Vafa] (1996)

$$Z(\tau) \sim \mathrm{Tr}\left[e^{2\pi i \tau L_0}\right] \sim \exp\left[\frac{\pi i c}{12\tau}\right] \quad \text{at } \tau \to i0^+$$

$$e^{S(P,c)} = \oint d\tau Z(\tau) e^{-2\pi i \tau P} \sim \exp\left[2\pi\sqrt{\frac{cP}{6}}\right] \quad \text{at } P \gg c \qquad \text{at } c = 6Q_1Q_5$$

- Derives the Bekenstein-Hawking entropy $S_{\rm BH} = 2\pi\sqrt{Q_1Q_5P}$.
- Many other BH's engineered and studied this way.

- Some BH's are simple, but some are complicated:
- single-centered, multi-center, black rings, "microstate geometry" , "BH with hairs", …
- Some BH's are dominant saddle points in non-Cardy regimes.
- Often uses different ad hoc descriptions for different BH's in a given system.

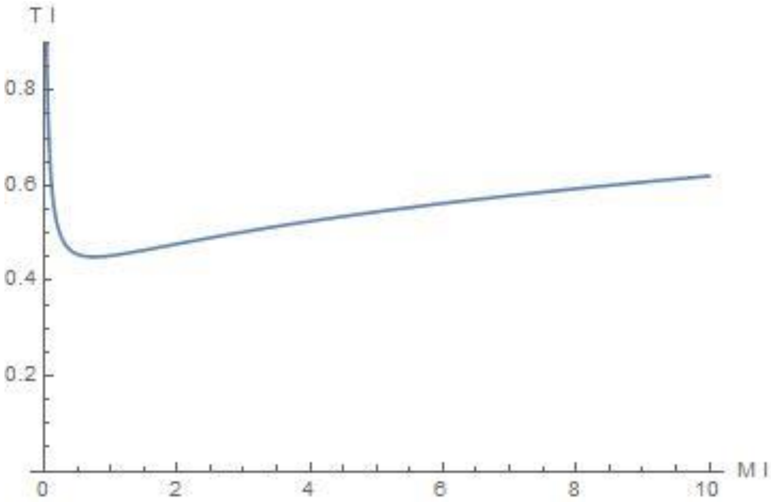
- Subjects of today's talk:
- Uniform description of many BH's in a given system: AdS black holes from CFT
- Establish a version of "Cardy formula" for the indices of $\mathrm{SCFT_D}$ in $D > 2$.

Black holes in AdS/CFT

- Schwarzschild black holes in global AdS_5:
- Exist small BH & large BH branches

$$T = \frac{r_+}{\pi \ell^2} + \frac{1}{2\pi r_+} \qquad r_+^2 = -\frac{\ell^2}{2} + \ell\sqrt{\frac{\ell^2}{4} + \omega M}$$

$$\omega \equiv \frac{16\pi G_N}{3\text{vol}(S^3)}$$



- Hawking-Page transition (1983): at $T = \frac{3}{2\,\pi\ell}$
- Low T : gas of gravitons in AdS. Doesn't see $1/\mathrm{G_N} \sim N^2$ so that $F \sim O(N^0)$
- High T : large AdS black holes ($F_{BH} = -T\log Z_{BH} < 0$). Sees N^2.

- CFT dual (on $S^3 \times R$): confinement-deconfinement transition [Witten] (1998)
- Confined phase: $F \sim O(N^0)$, glue-balls (& mesons, etc.)
- Deconfined phase: $F \sim O(N^2)$ of gluons (& quarks) ~ matrices

- Weak coupling study [Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk] (2003)
- Goal: Quantitative study of BPS BH's in $AdS_5 \times S^5$ from 4d $N = 4$ SYM

Supersymmetric black holes

- BPS black hole solutions [Gutowski,Reall] [Chong,Cvetic,Lu,Pope] [Kunduri,Lucietti,Reall]
- BPS energy (mass) determined by $U(1)^3 \subset SO(6)$ electric charges (momenta on S^5) & $U(1)^2 \subset SO(4)$ momenta on AdS_5 : $E\ell = Q_1 + Q_2 + Q_3 + J_1 + J_2$
- 2 real SUSY (1/16-BPS states). Very complicated solutions.

- Needs both Q_I & J_i to form BPS black holes.
- Otherwise, not enough microstates for form BH's.

- Charge relations: May naively expect 5 parameter solution, w/ 5 charges.
- But only finds solutions w/ 4 independent parameters [Kunduri, Lucietti, Reall] (2006)

- Recent numerical studies on more general hairy BH's in $AdS_5 \times S^5$.
- Hairs outside horizon. No charge relations [Markeviciute,Santos] [Bhattacharyya,Minwalla,Papa]

- The full set of BPS BH saddle points are probably unknown. (← from QFT later)

SUSY black holes from QFT

- Study the degeneracy of BPS states on $S^3 \times R$.
- More easily, study a "Witten index" type partition function on $S^3 \times R$,
- At the risk of losing certain states by boson/fermion cancelation from $(-1)^F$.

- This index was defined in [Romelsberger] [Kinney,Maldacena,Minwalla,Raju] (2005):

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\sum_{I=1}^{3} \Delta_I Q_I - \sum_{i=1}^{2} \omega_i J_i}\right]$$

$$\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 0$$

- Matrix integral expression for $\mathrm{U(N)}, \mathcal{N} = 4$ SYM (QFT on $S^3 \times S^1$) [KMMR] (2005) :

$$Z = \frac{1}{N!} \int \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \prod_{a<b} \left(2\sin\frac{\alpha_{ab}}{2}\right)^2 \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \frac{\prod_{I=1}^{3} 2\sinh\frac{n\Delta_I}{2}}{2\sinh\frac{n\omega_1}{2} \cdot 2\sinh\frac{n\omega_2}{2}}\right) \sum_{a,b=1}^{N} e^{in\alpha_{ab}}\right]$$

- Questions:
- Does the low "temperature" index agree w/ that of gravitons in $AdS_5 \times S^5$?
- Does this index undergo a deconfinement transition at certain $O(1)$ temperature?

Large N index

- Large N matrix integral → eigenvalue distribution:
- Integral variables α_a are $U(1)^N \subset U(N)$ holonomies on temporal circle.
- Identical particles on S^1, $\theta \sim \theta + 2\pi$: densely distributed at large N.
- Replaced at large N by a functional integral over their distribution,

$$\rho(\theta) = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\rho_n e^{in\theta} + \rho_{-n}e^{-in\theta}\right] \quad , \quad \rho_{-n} = \rho_n^*$$

$$\int_0^{2\pi} d\theta\rho(\theta) = 1$$

$$\rho(\theta) \geq 0$$

- From fine-grained picture:

$$\rho(\theta) = \frac{1}{N}\sum_{a=1}^{N}\delta(\theta - \alpha_a) = \frac{1}{2\pi N}\sum_{n=-\infty}^{\infty}\sum_{a=1}^{N} e^{in(\theta-\alpha_a)}$$

- Large N index: [Aharony,Marsano,Minwalla,Papadodimas,Raamsdonk] (2003) [KMMR] (2005)

$$Z = \int \prod_{n=1}^{\infty}[d\rho_n d\rho_{-n}] \exp\left[-N^2\sum_{n=1}^{\infty}\frac{1}{n}\rho_n\rho_{-n}\frac{\prod_I(1-e^{-n\Delta_I})}{\prod_i(1-e^{-n\omega_i})}\right]$$

- Low / high "temperature" ~ large/small Δ_I, ω_i (all being positive).
- "Low T": Saddle point at uniform distribution $\rho(\theta) = 1/2\pi$. "Confining phase"
- The confining index agrees w/ BPS graviton index on $AdS_5 \times S^5$ [KMMR] (2005).

Deconfinement from index?

- Does this index deconfine at high enough T?

$$Z = \int \prod_{n=1}^{\infty} [d\rho_n d\rho_{-n}] \exp\left[-N^2 \sum_{n=1}^{\infty} \frac{1}{n} \rho_n \rho_{-n} \frac{\prod_I (1-e^{-n\Delta_I})}{\prod_i (1-e^{-n\omega_i})}\right]$$

- Apparently, no, as the coefficients of Gaussian integrals are always positive,

$$f(\Delta_I, \omega_i) \equiv \frac{\prod_{I=1}^{3}(1-e^{-\Delta_I})}{\prod_{i=1}^{2}(1-e^{-\omega_i})} \qquad Z_{N\to\infty} = \prod_{n=1}^{\infty} f(n\Delta_I, n\omega_i)^{-1} = Z_{\text{gravitons}}$$

- … at real fugacities.
- So the index never seems to deconfine: Never sees a free energy at order $\log Z \sim N^2$.

- Apparent reason for this may be a severe B/F cancelation due to $(-1)^F$.
- 1st possibility: "microscopic cancelations" All coefficients suffer from cancelations

$$Z(x) = \sum_j \Omega_j x^j \qquad \log \Omega_j \sim \mathcal{O}(N^0) \text{ at } j \sim \mathcal{O}(N^2)$$

- 2nd possibility: "macroscopic cancelation" or smearing (→ next 2 slides)

Macroscopic index

- Unrefined index, for simplicity. $\Delta_1 = \Delta_2 = \Delta_3 \equiv \Delta,\ \omega_1 = \omega_2 \equiv \omega \qquad e^{-\omega} = x^3,\ e^{-\Delta} = x^2$

- E.g. U(2) index:

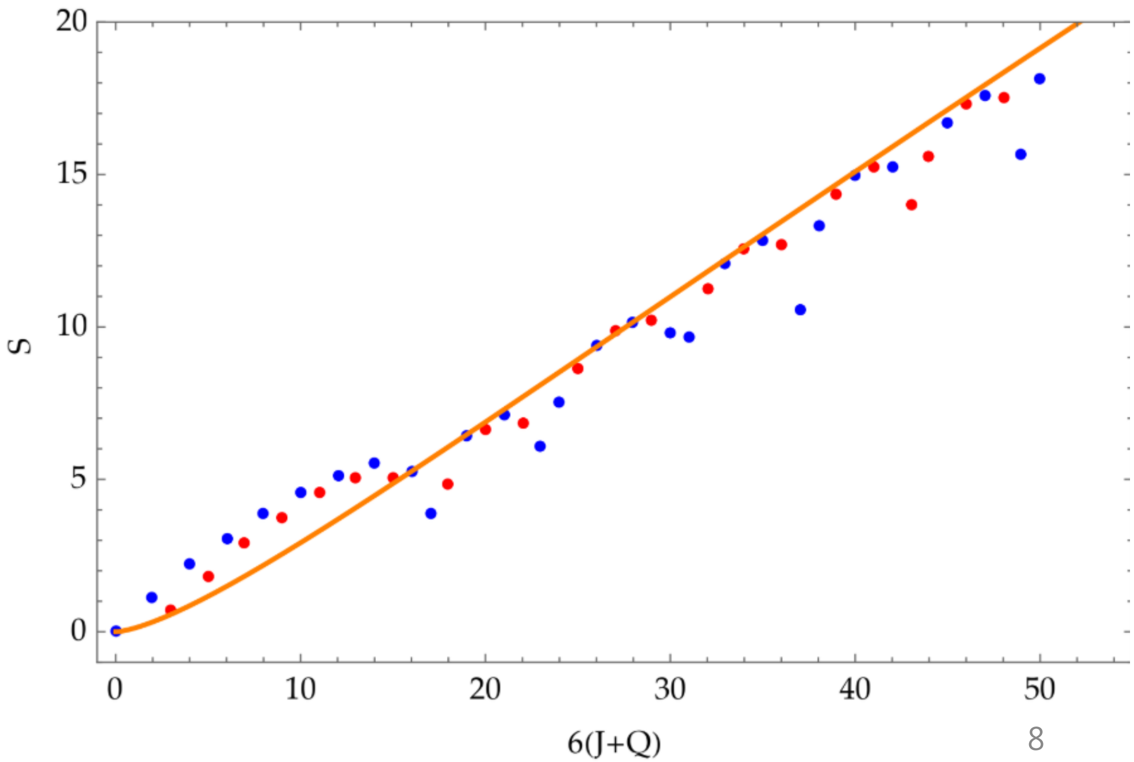
$$1 + 3x^2 - 2x^3 + 9x^4 - 6x^5 + 11x^6 - 6x^7 + 9x^8 + 14x^9 - 21x^{10} + 36x^{11} - 17x^{12} - 18x^{13}$$
$$+114x^{14} - 194x^{15} + 258x^{16} - 168x^{17} - 112x^{18} + 630x^{19} - 1089x^{20} + 1130x^{21} - 273x^{22}$$
$$-1632x^{23} + 4104x^{24} - 5364x^{25} + 3426x^{26} + 3152x^{27} - 13233x^{28} + 21336x^{29} - 18319x^{30}$$
$$-2994x^{31} + 40752x^{32} - 76884x^{33} + 78012x^{34} - 11808x^{35} + \cdots .$$

- Ω_j grows at large charges, but w/ sign oscillations.

- $\log|\Omega_j|$ for U(5).

$N^2 = 25 \gg 1 \ldots ?$

calculus/plot by [Agarwal, Nahmgoong]



orange: $S_{BH}(j)$ of known black hole solution, inserting $N^2 \to 25$
blue (B)/**red (F)** dots: $\log|\Omega_j|$ from the index

Macroscopic cancelation & obstructing it

- Consider a large charge $j \sim O(N^2)$ approximation of $\Omega_j = \frac{1}{2\pi i}\oint \frac{dx}{x^{j+1}} Z(x)$

- Saddle pt. calculus (Legendre transform): insensitive to changing j by a quantum
- What should we expect to be the result of this macroscopic approximation?

- 1st possibility: Unable to capture single Ω_j w/ wild oscillation. "Smears out" nearby Ω_j's.
- 2nd possibility: Can we get macroscopic entropy w/ wild ± 1 oscillation from phase factor?

  Namely, $\Omega_j \sim e^{S(j,x_*)} = e^{Im[S(j,x_*)]}\, e^{Re[S(j,x_*)]}$

- For the latter possibility to happen, one should turn on complex fugacities, because the idea is to realize +/- oscillations by a rapidly rotating phase.

- In a different perspective, the fugacity phase is to be tuned,
- attempting to tame rapid oscillations between +/- signs,
- or to maximally obstruct cancelations (smearing) of nearby B/F.

Upper bound on deconfinement

- Large N index with fugacity phases.

- Again, unrefine $\Delta_1 = \Delta_2 = \Delta_3 \equiv \Delta,\ \omega_1 = \omega_2 \equiv \omega$: $e^{-\omega} = x^3,\ e^{-\Delta} = x^2$

- The index w/ $x \to x\, e^{i\phi}$

$$Z = \int \prod_{n=1}^{\infty} [d\rho_n d\rho_{-n}] \exp\left[-N^2 \sum_{n=1}^{\infty} \frac{f(x^n)}{n} \rho_n \rho_{-n}\right] \qquad f(x) = \frac{(1-x^2)^3}{(1-x^3)^2}$$

- Search for condensation of ρ_1, dialing ϕ : $\rho_1 = 0$ is unstable if $Re\left[f\left(xe^{i\phi}\right)\right] < 0$.

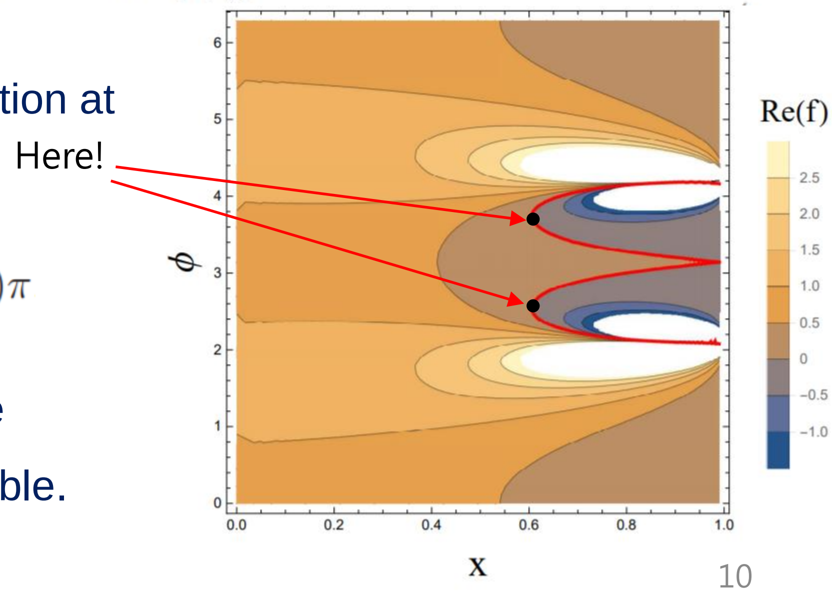
- Depending on ϕ, smearing may be partly obstructed, so that phase transition may be less delayed. Seek for the least delayed transition at optimized ϕ.

- $Re[f(xe^{i\phi})]$: $\dfrac{(1-x^2)(1+x^2-2x\cos\phi)^2\,(2x(2+5x^2+2x^4)\cos\phi + (1+x^2)(1+4x^2+x^4+3x^2\cos(2\phi)))}{(1+x^6-2x^3\cos(3\phi))^2}$

- red curve: $Re\left[f\left(xe^{i\phi}\right)\right] = 0$. Least delayed transition at

$$x_H = \sqrt{\frac{\sqrt{3}-1}{2}} \approx 0.605 \qquad \cos\phi = -\frac{1}{2x_H}$$

$$\phi \approx 0.81\pi \ \text{or} \approx (2-0.81)\pi$$



- Sets an upper bound of deconfinement, because the confining saddle point becomes locally unstable.

Cardy limit

- We shall first study a simple sector: "high temperature limit"
- large spin limit ~ high T limit, $|\omega_i| \ll 1$.
- higher D analogue of 2d Cardy formula at $|\tau| \ll 1$

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\sum_{I=1}^{3}\Delta_I Q_I - \sum_{i=1}^{2}\omega_i J_i}\right]$$

- Already studied for the 4d index [Di Pietro, Komargodski] [Ardehali] ……
- But only at real fugacities. Don't see macroscopic free energy for BH's.
- E.g. finds following subleading term after partial cancelation [Di Pietro, Komargodski] (2014)

$$-\log Z \sim \frac{a-c}{\beta} \qquad\qquad \beta \sim \omega_{1,2} \ll 1$$

- Large charge (~ BPS energy) means small chemical potentials: "real parts"
- Keep O(1) imaginary parts of chemical potentials for internal symmetries.
- Convenient to change convention for the chemical potentials:

From:

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\sum_{I=1}^{3}\Delta_I Q_I - \sum_{i=1}^{2}\omega_i J_i}\right]$$

$$\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 0$$

→

To:

$$\mathrm{Tr}\left[e^{-\sum_{I=1}^{3}\Delta_I Q_I - \sum_{i=1}^{2}\omega_i J_i}\right]$$

$$\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2\pi i \pmod{4\pi i}$$

Should optimally distribute $2\pi i$ (caused by $(-1)^F$)
to chemical potentials, minimizing B/F cancelations

Generalized Cardy limit

- Matrix integral expression in the new basis:

$$\frac{1}{N!}\oint\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot\prod_{a<b}\left(2\sin\frac{\alpha_{ab}}{2}\right)^2\exp\left[\sum_{a,b=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}\left(1+\sum_{s_1,s_2,s_3=\pm1}\frac{s_1s_2s_3(-1)^{n-1}e^{\frac{ns_I\Delta_I}{2}}}{2\sinh\frac{n\omega_1}{2}\cdot 2\sinh\frac{n\omega_2}{2}}\right)e^{in\alpha_{ab}}\right]$$

- $|\omega_i| \ll 1$ approximation. (O(1) imaginary part for all Δ_I's)

$$Z\sim\frac{1}{N!}\oint\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\exp\left[-\frac{1}{\omega_1\omega_2}\sum_{a\neq b}\sum_{s_1,s_2,s_3=\pm1}s_1s_2s_3\mathrm{Li}_3\left(-e^{\frac{s_I\Delta_I}{2}}e^{i\alpha_{ab}}\right)\right]$$

$$\mathrm{Li}_3(x)\equiv\sum_{n=1}^{\infty}\frac{x^n}{n^3}$$

- "Maximally deconfining" saddle $\alpha_1=\alpha_2=\cdots=\alpha_N$ is most dominant [CKKN] [Honda] [Ardehali]

- Result: [Choi, Joonho Kim, SK, Nahmgoong] (2018)

Use: $\mathrm{Li}_3(-e^x)-\mathrm{Li}_3(-e^{-x})=-\frac{x^3}{6}-\frac{\pi^2 x}{6}$

$$\log Z\sim-\frac{N^2}{\omega_1\omega_2}\sum_{s_1s_2s_3=+1}\left[\mathrm{Li}_3\left(-e^{\frac{s_I\Delta_I}{2}}\right)-\mathrm{Li}_3\left(-e^{-\frac{s_I\Delta_I}{2}}\right)\right]\xrightarrow{-\pi<\mathrm{Im}(x)<\pi}\log Z\sim\frac{N^2\Delta_1\Delta_2\Delta_3}{2\omega_1\omega_2}$$

- This is our Cardy free energy, valid at $|\omega_i| \ll 1$ and $\Delta_1+\Delta_2+\Delta_3-\omega_1-\omega_2=2\pi i$
- Explained later [Benini, Milan] that it holds beyond Cardy limit, at "certain" local saddle pt.

Macroscopic entropy

- So far, we established the Cardy formula for the index of 4d U(N) MSYM
- In the grand canonical ensemble.
- Macroscopic free energy, $\propto N^2$ as we further take a large N limit.

- To compute the macroscopic entropy, go to the microcanonical ensemble.
- inverse Laplace transform: large charge approx. by Legendre transform

$$S(\Delta_I, \omega_i; Q_I, J_i) = \frac{N^2}{2}\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} + \sum_{I=1}^{3} Q_I\Delta_I + \sum_{i=1}^{2} J_i\omega_i \qquad \Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2\pi i$$

- This extremization problem was discussed in [Hosseini, Hristov, Zaffaroni] (2017), from gravity.
- We have just seen a QFT derivation.

- $S(Q_I, J_i)$ at the saddle pt. is in general complex.
- $Im(S)$: Should be $\in \pi Z$ after extremization, from unitarity of the QFT: $e^{i\, Im(S)} = \pm 1$.
- One can argue this indirectly based on $4\pi i$-periodicities of Δ_I, ω_i.
- $Re(S)$: We will count known BH's by confirming $Re(S) = S_{BH}$.

Counting large black holes

- Here, recall that known BPS BH's satisfy a charge relation.
- To see if our index counts these BH's, impose this relation by hand.

- Results: [Choi, Joonho Kim, SK, Nahmgoong]

$$
\begin{aligned}
S(Q_I, J_i) &= 2\pi\sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1 + J_2)} \\
&= 2\pi\sqrt{\frac{Q_1Q_2Q_3 + \frac{N^2}{2}J_1J_2}{\frac{N^2}{2} + Q_1 + Q_2 + Q_3}}
\end{aligned}
$$

known expression for S_{BH} [Kimyeong Lee, SK] (2006)

Compatibility of two expressions: charge relation of known BH's

- Accounts for large BPS BH's in $AdS_5 \times S^5$ from dual QFT, as the dominant saddle point.

- Extremization of $S(\Delta_I, \omega_i; Q_I, J_i)$ accounts for known BH's even away from Cardy limit.
- Similar to D1-D5-P BH's: $S_{\rm BH} = 2\pi\sqrt{Q_1Q_5P}$ even away from the Cardy limit
- Just local large N saddle point away from Cardy limit. (next slides)

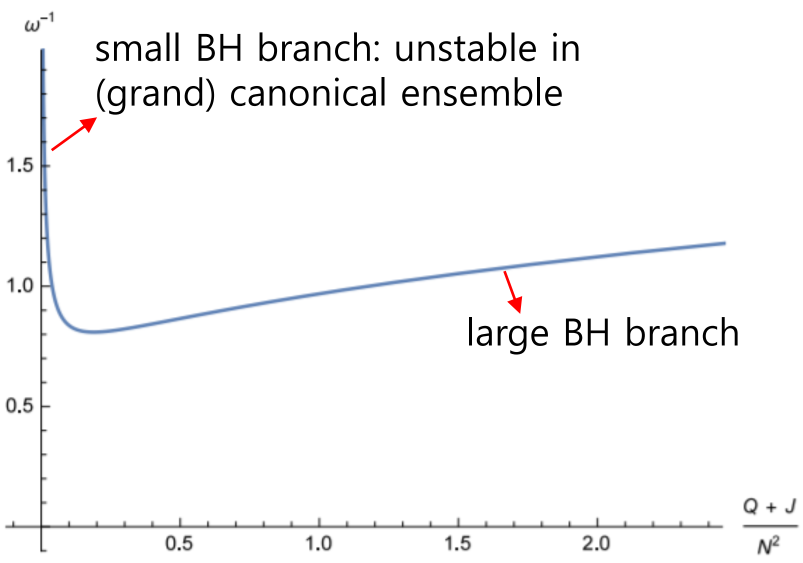
Away from large BH's

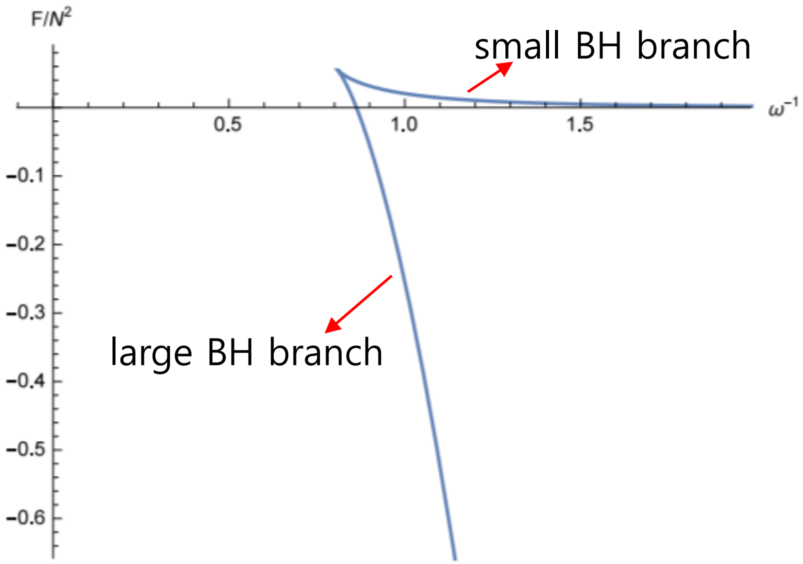
- Away from large BH limit, is the known BH solutions still the dominant saddles?

- For simplicity, set $Q \equiv Q_1 = Q_2 = Q_3$, $J \equiv J_1 = J_2$. Or, $\Delta \equiv \Delta_1 = \Delta_2 = \Delta_3$, $\omega \equiv \omega_1 = \omega_2$.
- Index: $3\Delta = 2\omega + 2\pi i$ admits one chemical potential ω, conjugate to $2(Q+J)$ in the index.

- Results: (similar to AdS Schwarzschild…!) Here, ω denotes $Re(\omega)$.

plays the role of T^{-1}, in the BPS sector



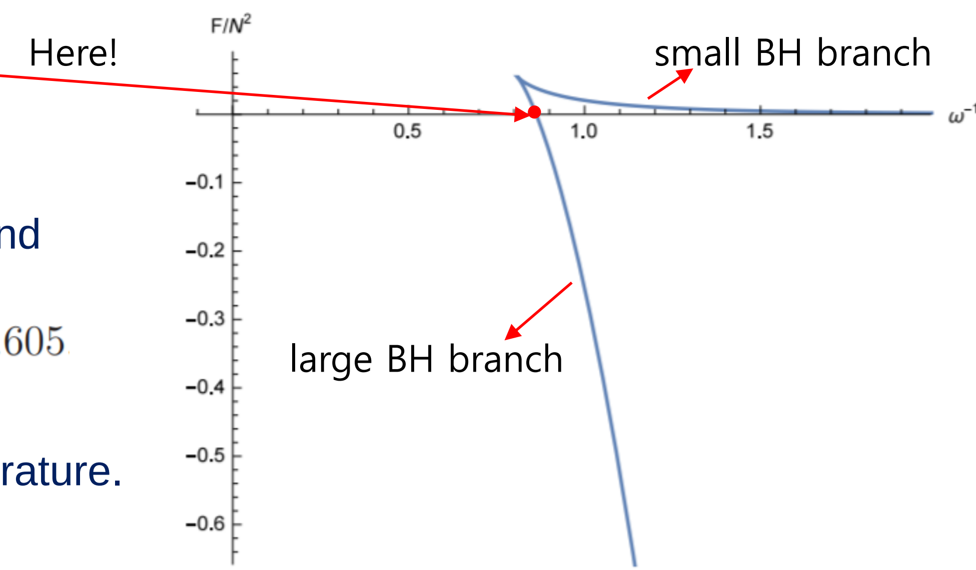


Hawking-Page transition of known BH's

- In canonical ensemble, many large N saddle points compete at given $T \sim \omega^{-1}$.
- One saddle point is the thermal graviton. $F \sim O(N^0)$
- There is also a black hole saddle point, given by known analytic solutions. $F \sim O(N^2)$
- There could possibly be more. (We shall see right below that they SHOULD exist.)

- If the known solutions are all BPS BH's, the competition of these & graviton saddles will decide the Hawking-Page transition.
- Transition "temperature" for known BH saddle point:

$$\omega_{\rm HP}^{\rm known} \equiv \frac{\pi}{16}\sqrt{414 - 66\sqrt{33}} \approx 1.16$$



- $(\omega_{HP}^{known})^{-1}$ is higher than our upper bound

$$\omega_{\rm H}^{-1} \approx 1.508^{-1} \qquad x_H = \sqrt{\frac{\sqrt{3}-1}{2}} \approx 0.605$$

- Index should deconfine below this temperature.
- This probably predicts new BH's.

Conclusion & comments

- Index of MSYM sees BPS AdS_5 black holes.
- Known large BH's in Cardy limit. Beyond this limit, new BH's should exist & may dominate.
- Still, local saddle points of known BH's are identified [Benini, Milan]
- More evidences for new BH's: Studied a 1/8-BPS subsector of MSYM ("Macdonald index"), in which known BH's cease to exist, and found new BH-like saddle points. [CKKN] 1810

- Further works: (some to appear soon)
- Based on 't Hooft anomaly: derived Cardy free energy of 6d (2,0). Counts large BPS BH's in $AdS_7 \times S^4$. [CKKN] 1810, [Nahmgoong]

$$\log Z \sim -\frac{N^3}{24}\frac{\Delta_1^2\Delta_2^2}{\omega_1\omega_2\omega_3} - \frac{N}{192}\frac{((\Delta_1+\Delta_2)^2+4\pi^2)((\Delta_1-\Delta_2)^2+4\pi^2)}{\omega_1\omega_2\omega_3}$$

- BPS BH's in: AdS_4 /CFT_3 [Choi, Hwang, SK] in preparation & AdS_6 /CFT_5 [Choi, SK] to appear
- 4d $N = 1$ Cardy limit: macroscopic entropy in Hofman-Maldacena bound $1/2 < a/c < 3/2$. [Joonho Kim, SK, Jaewon Song] to appear

$$\log Z \sim (5a-3c)\frac{8\mu^3}{27\omega_1\omega_2} - (c-a)\frac{8\pi^2\mu}{3\omega_1\omega_2}$$

$$2\mu - \omega_1 - \omega_2 = 2\pi i$$

- Some numerical studies [Agarwal, SK, Nahmgoong] in some progress
- Some ideas on hairy black holes…