TBA equations and resurgent Quantum Mechanics

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March 13, 2019

Based on the work with Katsushi Ito and Marcos Mariño
1811.04812+work in progress

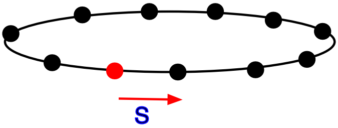
Classical and quantum integrable models

1. $\#$ conserved charges $=$ Degree of freedom of the system.
2. The system can be solved exactly.

Classical integrable model : KdV equation for soliton, affine Toda field equation, non-linear sigma model in AdS background.

Quantum Integrable model : XXX, XXZ, XYZ Heisenberg spin chain, quantum sine Gordon model, $\mathcal{N} = 4$ SYM in planar limit.

Bethe ansatz equation: the standard method to solve the spectrum of the quantum integrable model in the infinite volume.



Funtional relation and TBA equation in integrable model

- Thermodynamic Bethe ansatz (TBA) equations: powerful method to describe the thermodynamics of integrable model [Yang-Yang '69 Zamolodchikov '90].

- The TBA equations appear in the form of non-linear integral equations with good analytic properties.

- The TBA enables us to compute ground state energy (or effective charge of underlying CFT) of the integrable model in finite volume.

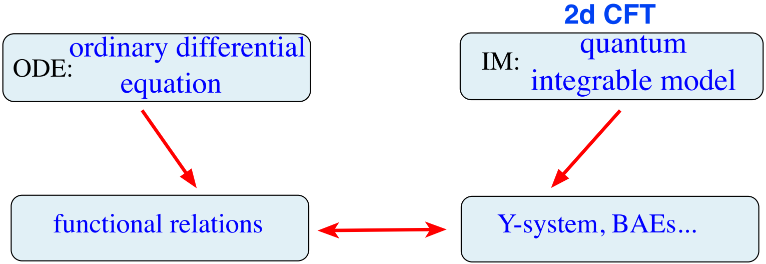
- TBA $\leftrightarrow$ Y-system + asymptotic behavior (functional relations classified in terms of (G, G')).

The same functional relations and the TBA equations also appear in some surprising areas, e.g. ODE's spectrum analysis: ODE/IM correspondence.

ODE/IM Correspondence

ODE: ordinary differential equation   IM: quantum integrable model (Integrals of motion)

This correspondence is based on the functional relations (Y-system, BAEs, $\cdots$) that appear on both sides.



Same form and same asymptotic behavior $\Longrightarrow$ Same solution.

Schrödinger type ODE [Dorey-Tateo'98, BLZ'98]

ODE		IM (2d CFT)
$\left(-\frac{d^2}{dz^2}+\frac{\ell(\ell+1)}{z^2}+z^{2M}-E\right)\psi=0$	$\Longleftrightarrow$	minimal model $M_{2,2M+2}$ (Six vertex model)

Functional relations: same form and same asymptotic behavior.

Integral equation of stokes multipliers $\Longleftrightarrow$ TBA equation

energy E $\Longleftrightarrow$ spectral parameter

level k	E_k (ODE)	E_k (TBA) [Dorey-Tateo '98]
0	1.0603620904841828	1.06036209048418
1	3.799673029801394	3.79967302980139
2	7.455697937986739	7.45569793798672

Remarks

1. The ODE/IM correspondence has been generalized to the higher order ODE [Dorey et.al. 1998-2007] and integrable nonlinear PDE [Lukyanov et.al 2010$\sim$].
2. No conceptual explanation is found so far, even in the simplest case.
3. We need to find more examples to understand the principle.
4. The potential z^{2M} is too special.

Motivation

- Understand the principle of ODE/IM correspondence.
- Generalize to general polynomial potential for more applications.
- Apply the ODE/IM correspondence (or integrability method) to gauge theory and AdS/CFT correspondence.
- Find the principle to study the non-perturbative physics through the simple example.

Schrödinger eq with general polynomial potential

$$V(x) = x^{r+1} - \sum_{a=0}^{r} u_a x^{r-a}$$

$$\left(-\hbar^2 \frac{d^2}{dx^2} + V(x) - E \right) \psi(x) = 0$$

→ Schroedinger equation in Quantum mechanics

→ ODE/IM correspondence

→ Quantum Seiberg-Witten curve of Argyres-Douglas th

Generalized ODE/IM correspondence

- A natural generalization of Dorey-Tateo's example:

$$\left(-\partial_z^2 + z^{r+1} + \sum_{a}^{r} b_a z^{r-a}\right)\psi(z) = 0.$$

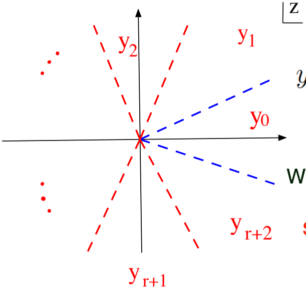
- Under the Symanzik rotation $(z, b_a) \to (\omega z, \omega^{a+1} b_a)$ with $\omega = e^{\frac{2\pi i}{r+3}}$, the ODE is invariant [Sibuya '75]

$$\omega^{-2}\left(-\partial_z^2 + \omega^{r+3} z^{r+1} + \omega^{r+3}\sum_{a=1}^{r} b_a z^{r-a}\right)\psi(\omega z, \omega^{a+1} b_a) = 0.$$

- If $\psi(z, b_a)$ is a solution of the ODE, $\psi(\omega z, \omega^{a+1} b_a)$ is also the solution of the ODE.

WKB approximation at large z

1. At large $|z|$, we solve the ODE using WKB approximation.



$$y_0(z, b_a) \sim \frac{1}{\sqrt{2i}} z^{n_r} \exp\left(-\frac{2}{r+3} z^{\frac{r+3}{2}}\right)$$

$|z| \to \infty$

We introduce the fastest decay solution at infinity in

sector k $\left|\arg(z) - \frac{2\pi k}{r+3}\right| < \frac{\pi}{r+3}$ as y_k

1. Stokes phenomenon.
2. There are $r + 3$ sectors on the complex plane.
3. $y_k(z, b_a) = \omega^{k/2} y_0(\omega^{-k} z, \omega^{-(a+1)} b_a)$
4. No singularity at $z = 0$, $y_{r+3}(z) \propto y_0(e^{-i2\pi} z) = y_0(z)$.

Y-function

- Wronskian $W_{i,j} = y_i \partial_z y_j - y_j \partial_z y_i = \text{const} \implies (y_0, y_1)$: a set of basis. The Wronskian plays the role of Stokes multiplier, i.e. the connection between the decaying solution in different sectors.

$$y_k = -\frac{W_{1,k}}{W_{0,1}} y_0 + \frac{W_{0,k}}{W_{0,1}} y_1$$

Y-function ($s \in \mathbb{Z}_{\geq 0}$)

$$\mathcal{Y}_{2j}(b_a) := \frac{W_{-j,j} W_{-j-1,j+1}}{W_{-j-1,-j} W_{j,j+1}}(b_a)$$

$$\mathcal{Y}_{2j+1}(b_a) = \frac{W_{-j-1,j} W_{-j-2,j+1}}{W_{-j-2,-j-1} W_{j,j+1}}(\omega^{\frac{a+1}{2}} b_a).$$

Y-system

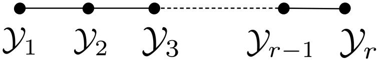
The Plücker relation, e.g.

$$W_{k_1+1,k_2+1} W_{k_1,k_2} = -W_{k_1+1,k_2} W_{k_2+1,k_1} - W_{k_1+1,k_1} W_{k_2,k_2+1}$$

yields the Y-system

$$\mathcal{Y}_s(\omega^{\frac{a+1}{2}} b_a)\mathcal{Y}_s(\omega^{-\frac{a+1}{2}} b_a) = \Big(1 + \mathcal{Y}_{s-1}(b_a)\Big)\Big(1 + \mathcal{Y}_{s+1}(b_a)\Big).$$

- $\mathcal{Y}_0 = 0$ by definition
- $y_0(z) \propto y_{r+3}(z) \to \mathcal{Y}_{r+1} = 0.$
- It is natural to truncate the Y-system at $\mathcal{Y}_r$. We thus obtain a A_r–type Y-system



- This Y-system is a straightforward generalization of the one in original ODE/IM correspondence where only $b_r \neq 0$.

- We have to control r parameters b_a in the same time to write the Y-system. It is difficult to write down the TBA equations.

- This is due to the complicated Symanzik rotation.

$$(\omega^{-k} z, \omega^{-(a+1)k} b_a) = \left((e^{i\pi k}\zeta)^{-\frac{2}{r+3}} x, -(e^{i\pi k}\zeta)^{-\frac{2(a+1)}{r+3}} u_a \right),$$

where we rescaled the parameters $x = \zeta^{\frac{2}{r+3}} z, -u_a = \zeta^{\frac{2(a+1)}{r+3}} b_a$

$$\left(-\zeta^2 \frac{d^2}{dx^2} + x^{r+1} - \sum_{a=1}^{r} u_a x^{r-a} \right) \hat{\psi}(x, u_a, \zeta) = 0.$$

Regarding ζ as spectral parameter

Solution of new equation: $\hat{y}(x, u_a, \zeta) = y(z, b_a)$

$$\hat{y}_k(x, u_a, \zeta) = \omega^{\frac{k}{2}} \hat{y}(x, u_a, e^{i\pi k}\zeta) = y_k(z, b_a)$$

$$\hat{W}_{j,i}(\zeta, u_a) = \zeta^{\frac{2}{r+3}} \det \begin{pmatrix} \hat{y}_j & \hat{y}_i \\ \partial_x \hat{y}_j & \partial_x \hat{y}_i \end{pmatrix} (\zeta, u_a) = W_{j,i}(b_a)$$

We also write the Y-function by using x and u_a such that ζ plays the role of spectral parameter. For later convenience, we relabel the index of the Y-function $s \to r + 1 - s$:

$$Y_{r+1-2j}(\zeta) = \frac{\hat{W}_{-j,j}\hat{W}_{-j-1,j+1}}{\hat{W}_{-j-1,-j}\hat{W}_{j,j+1}}(\zeta),\ Y_{r+1-2j-1}(e^{-\frac{\pi i}{2}}\zeta) = \frac{\hat{W}_{-j-1,j}\hat{W}_{-j-2,j+1}}{\hat{W}_{-j-2,-j-1}\hat{W}_{j,j+1}}(\zeta$$

Y-system in ζ [Alday et.al '2010, Ito-Marino-HS '18]

$$Y_s(e^{\frac{i\pi}{2}}\zeta) Y_s(e^{-\frac{i\pi}{2}}\zeta) = \Big(1 + Y_{s+1}(\zeta)\Big)\Big(1 + Y_{s-1}(\zeta)\Big),$$

Asymptotic behavior of Y-function

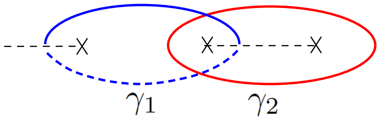
At small ζ, the WKB expansion of $\hat{y}_k(x, u_a, \zeta)$ is

$$\hat{y}_k(x, u_a, \zeta) = (-1)^{\frac{k}{2}} c(\zeta) \exp\left(-i\frac{\delta_k}{\zeta}\int_{x_k}^{x} Q(x')dx'\right).$$

The small ζ asymptotic behavior of $\log Y_s$:

$$\log Y_s \sim -i^{\frac{(-1)^s+1}{2}} \frac{1}{\zeta} \oint_{\gamma_s} dx \sqrt{E - V(x)} =: -\frac{m_s}{\zeta}, \quad |\arg(\zeta)| < \pi.$$

γ_s: one-cycle on the Riemann surface
$\xi^2 = V(x) - E = x^{r+1} - \sum_{a=1}^{r} u_a x^{r-a}$



Riemann surface for cubic case $(r = 2)$

TBA equations

Y-system + the asymptotic behavior → TBA equations
$(\zeta = e^{-\theta},\ m_s = |m_s| e^{i\phi_s},\ |\phi_{s+1} - \phi_s| < \frac{\pi}{2})$

$$\log Y_s(\theta - i\phi_s) = -|m_s| e^{\theta} + \int_{-\infty}^{\infty} \frac{\log[1 + Y_{s+1}(\theta' - i\phi_{s+1})]}{\cosh(\theta - \theta' - i\phi_s + i\phi_{s+1})} \frac{d\theta'}{2\pi}$$
$$+ \int_{-\infty}^{\infty} \frac{\log[1 + Y_{s-1}(\theta' - i\phi_{s-1})]}{\cosh(\theta - \theta' - i\phi_s + i\phi_{s-1})} \frac{d\theta'}{2\pi}, \quad s = 1, \cdots, r.$$

- These TBA equations coincide with the conformal limit of the TBA equations of A_r-type AD theory in the study of wall crossing [GMN '08, Gaiotto '14](up to mutation, wall crossing).

TBA in original ODE/IM

- Our TBA equations are quite different with the ones found by Dorey-Tateo.
- As $|\phi_{s+1} - \phi_s|$ cross $\frac{\pi}{2}$, we need to modify the TBA equations to pick up the contribution of pole in kernel(wall crossing of TBA).
- The ODE/IM of Dorey-Tateo, i.e. $b_1 = \cdots = b_{r-1} = 0$ and $b_r = E$, $r(r+1)/2 - r$ times wall crossing occur. Moreover, a Z_{r+1} symmetry enhances. We then recover the original TBA equations found by Dorey-Tateo.

Apply to the resurgent Quantum Mechanics

Conjecture: $\log Y \leftrightarrow$ WKB period [Ito-HS's 17]

At large θ, we expand the TBA equations:

$$\log Y_s(\theta) = -m_s e^{\theta} - \sum m_s^{(n)} e^{-(2n-1)\theta}$$

$$m_s^{(n)} = \frac{(-1)^n}{\pi} \int_{-\infty}^{\infty} \log \Big(1 + Y_{s+1}(\theta')\Big)\Big(1 + Y_{s-1}(\theta')\Big) d\theta'$$

The TBA equations can be solved numerically, from which we obtained the higher order correction of $\log Y_s$.

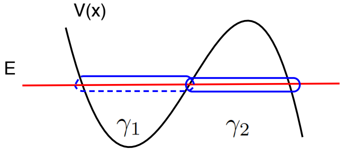
The WKB period (quantum period) in Quantum Mechanics is defined by

$$\frac{1}{\hbar}\Pi_s(\hbar) = \frac{1}{\hbar}\oint_{\gamma_s} Q(x;\hbar)dx = \frac{\Pi_s^{(0)}}{\hbar} + \sum \Pi_s^{(n)}\hbar^{2n-1},$$

by using which we could write down the quantization condition.

Let us consider the cubic case with the curve

$$V(x) = \frac{x^2}{2} - x^3$$



$E = 1/200$: TBA's result vs holomorphic anomaly equation's result [Ito-Marino-HS '18]

n	$\Pi_{\gamma_1}^{(n)}$	$m_1^{(n)}$	$\Pi_{\gamma_2}^{(n)}$	$m_2^{(n)}$
1	3.6574758326426	−3.6574758326423	−9.1939620228508	−9.1939620228505
2	948.794486716986	948.794486716980	19138.8317303045	19138.8317303047
3	1368408.366655312	−1368408.366655315	−228090464.089314	−228090464.089320

With high numerical check, one finds

$$m_1^{(n)} = (-1)^n \Pi_1^{(n)}, \quad m_2^{(n)} = i\Pi_2^{(n)}.$$

$$\log Y_1(\theta = \log(\frac{i}{\hbar})) = \frac{i}{\hbar}\Pi_{\gamma_1}(\hbar), \quad \log Y_2(\theta = \log(\frac{1}{\hbar})) = \frac{i}{\hbar}\Pi_{\gamma_2}(\hbar).$$

Suppose $\hbar$ is positive and real, there is a pole in the TBA equation of Y_1.

$$\log Y_1(\log \frac{1}{\hbar} + \frac{i\pi}{2} + i\delta) - \log Y_1(\log \frac{1}{\hbar} + \frac{i\pi}{2} - i\delta) = -\log Y_2(\log \frac{1}{\hbar})$$

This reproduces the discontinuities of the WKB periods obtained from the Borel resummation [Delabaere et.al '97].

Voros' Riemann–Hilbert problem

Voros' Riemann–Hilbert problem [Voros' 83]

Voros' Riemann–Hilbert problem: classical period and the discontinuity $\rightarrow$ Exact WKB periods.

Our TBA equations provide solution to Voros' Riemann–Hilbert problem:

Classical period : m_s
Discontinuity of Y_1 (or Π_{γ_1}) at positive real $\hbar$.

n	E_n^{num}	E_n^{TBA}(Our method)	$E_n^{\mathrm{BS,WKB}}$
0	1.156 267 071 988	1.156 267 071 988	1.094 269 500 533
1	4.109 228 752 810	4.109 228 752 806	4.089 496 119 273
2	7.562 273 854 979	7.562 273 854 971	7.548 980 437 586

Table: The energy levels for the PT cubic oscillator ($V(x) = ix^3$) for $\hbar = \sqrt{2}$. The first column shows a numerical calculation with complex dilatation techniques [Bender '07]. The second column is a calculation by using the exact quantization condition and the TBA system. The final column is the result of the Bohr–Sommerfeld approximation.

For higher degree polynomial potential, our TBA equations work perfectly as well.

Conclusion

- We have generalized the ODE/IM correspondence to the Schrödinger equaton with general polynomial potential.
- After several times wall crossing, our TBA equations recover the ones in the original ODE/IM.
- We found $\log Y \sim \Pi$, which allows us to compute the WKB periods (quantum periods) exactly.
- We found the solutions for Voros' Riemann–Hilbert problem. (classical period+ discontinuity $\rightarrow$ Exact WKB periods)

Discussion and Future work

- Potential $V(x)$ with more (ir)regular poles
  - Correlator of Wilson lines (loops) $\rightarrow$ non-planar scattering amplitude in AdS/CFT [Sever et al '18].
  - Quantum SW curve of $SU(N)$ gauge theory with N_f flavor.
  - Null vector decoupling equation $\rightarrow$ conformal block in classical limit, Painleve equation.
- The Riemann-Hilbert problem: boundary conditions+discontinuity $\rightarrow$ exact solution, also appear in many quantum system.
- Apply our logic to other more complicated physics ($\hbar \rightarrow$ coupling constant). To derive the TBA equations for coupling constant of QFT or String theory.

Thanks for your attention !